J. Fluid Mech. (1969), *vol.* 35, *part* 1, *pp.* 117–129
Printed in Great Britain

The degree of knottedness of tangled vortex lines

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(Received 17 May 1968)

Let $\mathbf{u}(\mathbf{x})$ be the velocity field in a fluid of infinite extent due to a vorticity distribution $\boldsymbol{\omega}(\mathbf{x})$ which is zero except in two closed vortex filaments of strengths κ_1, κ_2. It is first shown that the integral

$$I = \int \mathbf{u} . \boldsymbol{\omega} \, dV$$

is equal to $\alpha\kappa_1\kappa_2$ where α is an integer representing the degree of linkage of the two filaments; $\alpha = 0$ if they are unlinked, ± 1 if they are singly linked. The invariance of I for a continuous localized vorticity distribution is then established for barotropic inviscid flow under conservative body forces. The result is interpreted in terms of the conservation of linkages of vortex lines which move with the fluid.

Some examples of steady flows for which $I \neq 0$ are briefly described; in particular, attention is drawn to a family of spherical vortices with swirl (which is closely analogous to a known family of solutions of the equations of magnetostatics); the vortex lines of these flows are both knotted and linked.

Two related magnetohydrodynamic invariants discovered by Woltjer (1958*a*, *b*) are discussed in §5.

1. Introduction; discrete vortex fields

Consider any flow $\mathbf{u}(\mathbf{x}, t)$ under conservative body forces, of an inviscid fluid whose density is either uniform or a function of pressure only. Under these conditions, the circulation round any circuit C moving with the fluid,

$$K = \oint_C \mathbf{u} . d\mathbf{l}, \tag{1}$$

is constant.

In the particular circumstance that the vorticity $\boldsymbol{\omega} = \nabla \wedge \mathbf{u}$ is zero except inside two closed vortex filaments C_1, C_2 of strengths κ_1, κ_2 each of which moves with the fluid, we may choose C to be one of these, say C_1. If C_1 is unknotted, so that it can be spanned by a surface S_1 which does not intersect itself, then Stokes's theorem gives

$$K_1 = \oint_{C_1} \mathbf{u} . d\mathbf{l} = \int_{S_1} \boldsymbol{\omega} . d\mathbf{S}, \tag{2}$$

and so, since the flux of vorticity across S_1 is simply that due to the filament C_2,

$$K_1 = \begin{cases} 0 & \text{if } C_1 \text{ and } C_2 \text{ are not linked,} \\ \pm\kappa_2 & \text{if } C_1 \text{ and } C_2 \text{ are singly linked,} \end{cases}$$

(figure 1), the $\pm$ referring to the two possible relative orientations of the two filaments. More generally, the filament C_2 may wind an integral number of times round C_1 in which case

$$K_1 = \alpha_{12}\kappa_2, \tag{3}$$

where α_{12} ($= \alpha_{21}$) is an integer which may be positive or negative (the 'winding number' of the curves C_1 and C_2).†

A simple vortex line that is knotted may be decomposed into two (or more) linked but unknotted vortex lines by the insertion of a pair (or pairs) of equal and opposite vorticity segments. For example, if the vorticity field is zero except in a vortex filament of strength κ having the shape C in figure 2 (the trefoil knot), then

$$\oint_C \mathbf{u}.d\mathbf{l} = \oint_{C_1} \mathbf{u}.d\mathbf{l} + \oint_{C_2} \mathbf{u}.d\mathbf{l} = 2\kappa.$$

For a more complicated knot in a vortex filament C,

$$\oint_C \mathbf{u}.d\mathbf{l} = 2\alpha\kappa,$$

where α is an integer representing the degree of knottedness of C, the 'self-winding' number of C. All knots will be supposed in what follows to be dealt with in this manner.

If there are n unknotted filaments $C_1, C_2, \ldots, C_n$, then a simple generalization of the result (3) is

$$K_i = \oint_{C_j} \mathbf{u}.d\mathbf{l} = \sum_j \alpha_{ij}\kappa_j, \tag{4}$$

where α_{ij} is the winding number of C_i and C_j.

The quantity $\kappa_i K_i$ (not summed) may be written in the form of an integral over the volume V_i occupied by the vortex filament C_i. Since $d\mathbf{l}$ is parallel to $\boldsymbol{\omega}$ in the filmanet, $\kappa_i d\mathbf{l}$ may be replaced by $\boldsymbol{\omega}\,dV$ so that

$$\kappa_i K_i = \oint_{C_i} \kappa_i d\mathbf{l}.\mathbf{u} = \int_{V_i} \mathbf{u}.\boldsymbol{\omega}\,dV. \tag{5}$$

If we sum over all the filaments, we obtain an invariant integral over the whole vorticity field:

$$I = \sum_i \kappa_i K_i = \sum_{i,j} \alpha_{ij}\kappa_i\kappa_j = \int_V \mathbf{u}.\boldsymbol{\omega}\,dV, \tag{6}$$

† The term 'winding number' (anzahl der umschlingungen) and the expression given below for it, equation (11), can be traced to a paper by Gauss (1833) which was concerned with the magnetic field produced by two or more electric current circuits. It is the simplest (but by no means the only) topological invariant of two linked curves (see, for example, Crowell & Fox 1964, and the references given therein).

The possibility of linked and knotted vortex lines was conceived by Kelvin (1868, then Sir William Thomson) in his celebrated paper, on 'Vortex Motion', in which the 'circulation theorem' was established. The simplest knots were subsequently catalogued by Tait (1898, pp. 273–347) in increasing order of complexity. The development of knot theory as a recognizable branch of modern topology received considerable stimulus from these investigations.

where V is the volume occupied by all the filaments, or equivalently (as far as the integral in (6) is concerned) the total volume occupied by the fluid. It should be noted that I is determined solely by the vorticity field; this dependence may be made explicit by writing

$$\mathbf{u} = \mathbf{u}_1(\mathbf{x}) + \nabla\phi, \tag{7}$$

where

$$\mathbf{u}_1(\mathbf{x}) = -\frac{1}{4\pi}\int \frac{\mathbf{R}\wedge\boldsymbol{\omega}(\mathbf{x}')}{R^3}\,dV' \quad (\mathbf{R} = \mathbf{x}-\mathbf{x}'). \tag{8}$$

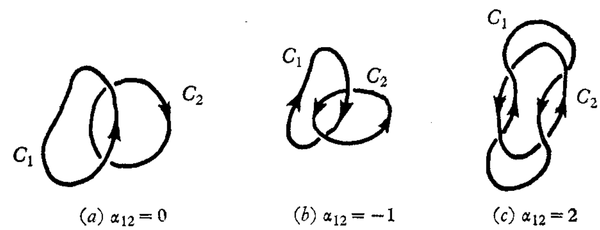

FIGURE 1. The degree of linkage of two closed filaments C_1, C_2. The choice of sign in (b), (c) is determined by the relative orientation of the two filaments.

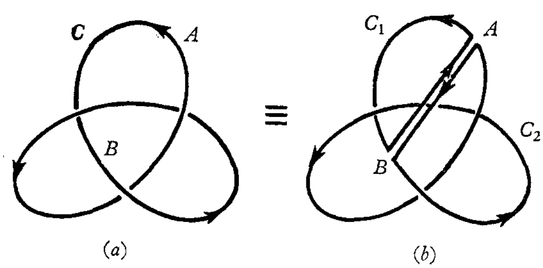

FIGURE 2. Decomposition of a knotted vortex line. To get from (a) to (b), two equal and opposite vorticity segments are inserted between the points A and B. C_1 and C_2 are evidently unknotted but linked.

The potential contribution to $\mathbf{u}$ (which is certainly present if the fluid is enclosed by a rigid boundary) makes no contribution to I since

$$\int_V \nabla\phi\,.\,\boldsymbol{\omega}\,dV = \int_V \nabla\,.\,(\boldsymbol{\omega}\phi)\,dV = \int_S \mathbf{n}\,.\,\boldsymbol{\omega}\phi\,dS = 0. \tag{9}$$

Substitution of (8) in (6) then gives

$$I = \frac{1}{4\pi}\iint \frac{\mathbf{R}\,.\,[\boldsymbol{\omega}(\mathbf{x})\wedge\boldsymbol{\omega}(\mathbf{x}')]}{R^3}\,dV\,dV'. \tag{10}$$

If this is re-expressed in terms of line integrals, we obtain an explicit expression for α_{ij} $(i \neq j)$ in terms of the relative geometry of the circuits C_i and C_j:

$$\alpha_{ij} = \frac{1}{4\pi}\oint_{C_i}\oint_{C_j} \frac{\mathbf{R}\,.\,d\mathbf{l}_i \wedge d\mathbf{l}_j}{R^3} = \alpha_{ji}, \tag{11}$$

where

$$\mathbf{R} = \mathbf{x}_i - \mathbf{x}_j, \quad \mathbf{x}_i \in C_i, \quad \mathbf{x}_j \in C_j.$$

2. Continuous vorticity fields

Consider now a continuous localized distribution of vorticity in an infinite expanse of inviscid fluid. In general the vortex lines will not be closed; a single vortex line may cover a surface,† or it may even pass arbitrarily close to any point of a closed volume, if followed far enough. (The instantaneous vorticity field in a turbulent 'blob' is likely to have this latter character in general.) The simple considerations of the preceding section are not therefore directly applicable. However, it might be expected that the integral I defined in (6), might still be an invariant for a continuous vorticity blob, and that, if so, it may give a useful generalization of the concept of 'degree of knottedness' to a continuous solenoidal vector field.

Let us first obtain an equation for the local rate of change of the quantity $\mathbf{u}.\boldsymbol{\omega}/\rho$. Under the barotropic condition $p = p(\rho)$, the equation of motion may be written

$$D\mathbf{u}/Dt = -\nabla(h+\Omega), \tag{12}$$

where $h = \int dp/\rho$ and Ω is the potential of any conservative body forces. Under the same conditions, the vorticity equation takes the well-known form

$$\frac{D}{Dt}\left(\frac{\boldsymbol{\omega}}{\rho}\right) = \frac{\boldsymbol{\omega}}{\rho}.\nabla\mathbf{u}. \tag{13}$$

Hence

$$\frac{D}{Dt}\left(\frac{\mathbf{u}.\boldsymbol{\omega}}{\rho}\right) = -\frac{\boldsymbol{\omega}}{\rho}.\nabla(h+\Omega)+\mathbf{u}.\left(\frac{\boldsymbol{\omega}}{\rho}.\nabla\right)\mathbf{u}$$

$$= \frac{\boldsymbol{\omega}}{\rho}.\nabla(\tfrac{1}{2}q^2-h-\Omega), \tag{14}$$

where

$$q^2 = \mathbf{u}.\mathbf{u}.$$

Now let S be any surface enclosing a volume V and moving with the fluid, and let

$$I = \int_V \mathbf{u}.\boldsymbol{\omega}\,dV. \tag{15}$$

Since

$$\frac{D}{Dt}(\rho\,dV) = 0, \tag{16}$$

it follows that

$$\frac{dI}{dt} = \int_V \frac{D}{Dt}\left(\frac{\mathbf{u}.\boldsymbol{\omega}}{\rho}\right)dV$$

$$= \int_V (\boldsymbol{\omega}.\nabla)(\tfrac{1}{2}q^2-h-\Omega)\,dV \qquad \text{from (14)}$$

$$= \int_S (\mathbf{n}.\boldsymbol{\omega})(\tfrac{1}{2}q^2-h-\Omega)\,dS, \tag{17}$$

† For example, if the flow is steady, and the body forces have a potential Ω, then

$$\boldsymbol{\omega}\wedge\mathbf{u} = \nabla H \quad \text{where} \quad H = \tfrac{1}{2}q^2+\Omega+\int dp/\rho,$$

and the $\boldsymbol{\omega}$-lines lie on the surfaces H = constant.

using $\nabla . \boldsymbol{\omega} = 0$ and the divergence theorem. Hence the condition $\mathbf{n} . \boldsymbol{\omega} = 0$ on S is sufficient to ensure that

$$I = \text{constant}.$$

If the fluid is of infinite extent, and S is taken to be the surface 'at infinity', then the condition $\boldsymbol{\omega} = o(R^{-4})$ as $R \equiv |\mathbf{x}| \to \infty$ is likewise sufficient to ensure the invariance of I.

If the vortex lines of the field $\boldsymbol{\omega}$ are all closed then there is a separate invariant I for each closed vortex filament in the field (the volume V in (15) being then the volume occupied by the filament). In the limit as the cross-section of each filament is decreased to zero, we have a doubly infinite family of invariants. If the vortex lines cover surfaces, then there is an invariant I for each 'vorticity layer' in the neighbourhood of each surface, and as the thickness of the layers is decreased to zero, we have a singly infinite family of invariants. If the vortex lines of the field are 'space-filling', then there is only one invariant I for each subdomain of V that is filled by a vortex line.

The quantity $\mathbf{u} . \boldsymbol{\omega}$ admits a simple, essentially kinematical, interpretation. The fluid particles in any small volume element dV undergo at any instant a superposition of three motions: the (uniform) velocity $\mathbf{u}_0$ of any representative point 0 of the element, an irrotational uniform strain $\nabla\phi$ relative to I, and a rigid body rotation $2\boldsymbol{\omega}_0$ where $\boldsymbol{\omega}_0$ is the vorticity at 0. The streamlines of the flow $\mathbf{u} - \nabla\phi$ passing near 0 are (locally) helices about the streamline through 0, and the contribution

$$\mathbf{u} . \boldsymbol{\omega} dV \approx \mathbf{u}_0 . \boldsymbol{\omega}_0 dV$$

to I from dV is positive or negative according as the screw of these helices is right-handed or left-handed. The term *helicity* is used in particle physics for the scalar product of the momentum and spin of a particle, and it would seem to be a natural candidate in the present context to describe the quantity $\mathbf{u} . \boldsymbol{\omega} dV$; the quantity $\mathbf{u} . \boldsymbol{\omega}$ may then be described as the *helicity per unit volume* of the flow. Equation (17) then expresses the result that the total helicity within any closed vortex surface (on which $\boldsymbol{\omega} . \mathbf{n} = 0$) is constant.

3. The effect of the presence of solid boundaries

An inviscid flow in the presence of a solid boundary S_b need not satisfy the condition $\mathbf{n} . \boldsymbol{\omega} = 0$ on S_b since $\mathbf{n} \wedge \mathbf{u}$ may vary from one point to another on the boundary. It would therefore appear that the value of I may then change according to (17), and this is at first sight surprising in view of the interpretation given above of the invariance of I in terms of the conservation of linkages of vortex lines; vortex lines are still frozen in the fluid when rigid boundaries are present, so these should not affect the invariance of I.

The explanation lies in the fact that if $\mathbf{n} . \boldsymbol{\omega} \neq 0$ on S_b and if S_b is at rest, then there exists a vortex sheet on S_b and the vortex lines of the fluid interior must be imagined to be continued and completed within this sheet. (If S_b is rotating, there is the further complication that the vortex lines actually continue into the solid.) We should therefore expect invariance of the quantity I defined in (15) only if it is supplemented by a finite contribution from the vortex sheet on the surface, and possibly a contribution from the interior of the solid surroundings.

The surface contribution depends on the structure of the vortex sheet on S_b. The thickness of the vortex sheet (or boundary layer) is controlled by viscous forces, and it is physically unrealistic to ignore these in any treatment of the surface layer. Suppose, for simplicity, that S_b is at rest, and that the fluid has a small kinematic viscosity ν. Then $\mathbf{u} = 0$ on S_b, so that $\boldsymbol{\omega}.\mathbf{n} = 0$ on S_b. However, we must now include viscous terms $\nu\nabla^2\mathbf{u}$ and $\rho^{-1}\nu\nabla^2\boldsymbol{\omega}$ on the right-hand sides of (12) and (13), and this leads to

$$\frac{dI}{dt} = -2\nu\int_V \boldsymbol{\omega}.(\nabla\wedge\boldsymbol{\omega})\,dV, \tag{18}$$

where I is still defined as in (15) (so that it now includes a 'surface contribution' distributed through the boundary layer) and V is the total volume occupied by the fluid.

Let L be the scale of variation of $\mathbf{u}$ in the tangential directions (on S_b), and let q_0 be the scale of $|\mathbf{u}|$ just outside the boundary layer. Then the thickness of the layer is (in general)

$$\delta = O(\nu^{\frac{1}{2}}L^{\frac{1}{2}}/q_0^{\frac{1}{2}}), \tag{19}$$

and the normal and tangential components of vorticity in the layer have orders of magnitude

$$\omega_n \equiv |\boldsymbol{\omega}.\mathbf{n}| = O(q_0/L), \quad \omega_s \equiv |\boldsymbol{\omega}\wedge\mathbf{n}| = O(q_0/\delta). \tag{20}$$

Hence

$$|\nu\boldsymbol{\omega}.\nabla\wedge\boldsymbol{\omega}| = O(\nu\omega_s^2/\delta) = O(\omega_n q^2/\delta|, \tag{21}$$

and so the contribution to dI/dt from unit area of the boundary layer is, from (18), of order $\omega_n q_0^2$ and this is independent of ν in the limit $\nu\to 0$. The structure of the boundary layer is therefore of critical importance as $\nu\to 0$ in determining not only the value of I, but also its rate of change dI/dt.†

4. A simple consequence of the invariance of *I* in an incompressible fluid

Henceforth we restrict attention to vorticity blobs in an inviscid incompressible fluid with $\boldsymbol{\omega}.\mathbf{n} = 0$ on all solid boundaries. The integrals

$$I = \int \mathbf{u}.\boldsymbol{\omega}\,dV, \quad E = \frac{2T}{\rho} = \int \mathbf{u}^2 dV, \quad \Omega = \int \boldsymbol{\omega}^2 dV, \tag{22}$$

satisfy the Schwarz inequality

$$I^2 \leqslant E\Omega, \quad \text{or} \quad \Omega \geqslant I^2/E, \tag{23}$$

† This may be contrasted (in the incompressible case) with the behaviour of the energy

$$T = \tfrac{1}{2}\rho\int \mathbf{u}^2 dV = \rho\int \mathbf{u}.(\mathbf{x}\wedge\boldsymbol{\omega})\,dV,$$

for a localized vorticity blob, which satisfies

$$\frac{dT}{dt} = -2\nu\int \boldsymbol{\omega}^2 dV,$$

when S_b is stationary. As $\nu\to 0$, $dT/dt = O(\nu^{\frac{1}{2}})$ and $T\to$ constant, independent of the boundary-layer structure.

and since both I and E are invariants, Ω has a fixed lower bound, which is evidently attained only if $\boldsymbol{\omega} = \alpha\mathbf{u}$ where α is constant.

To understand the physical significance of this result, consider the following situation. Suppose that a vortex ring propagates along an axisymmetric duct of decreasing cross-section, the axes of symmetry of the duct and of the ring coinciding. Evidently, the value of Ω for this vorticity field may become arbitrarily small if the radius of the duct becomes sufficiently small; but since $\mathbf{u}\,.\,\boldsymbol{\omega} = 0$ for this flow, $I = 0$ and there is no contradiction with (23). Suppose now instead that a blob† of vorticity for which $I \neq 0$ is so disposed as to propagate into a similar contraction; is it then physically conceivable that the value of Ω for the blob can be made to decrease without limit by choosing a contraction of suitable geometry? The answer is negative, consistently with (23), for the following reason. Since $I \neq 0$, there must exist knots or links in the vortex lines of the blob. No single Cartesian component of the vorticity field can then be identically zero (since curves confined to a plane cannot be knotted or linked). Since the volume of the blob is constant, any decrease in the components of vorticity perpendicular to the axis of the duct is then necessarily accompanied by an increase (through stretching) of the vorticity component parallel to the axis, and it is therefore evident that Ω cannot decrease indefinitely.

5. Relation with the magnetohydrodynamic invariants of Woltjer (1958*a*, *b*)

Let $\mathbf{B} = \nabla \wedge \mathbf{A}$ and $\mathbf{E} = -\partial\mathbf{A}/\partial t - \nabla\phi$ be the magnetic field and electric field in a perfectly conducting fluid; then, since $\mathbf{E} + \mathbf{u} \wedge \mathbf{B} = 0$,

$$\partial\mathbf{A}/\partial t = \mathbf{u} \wedge (\nabla \wedge \mathbf{A}) - \nabla\phi, \tag{24}$$

and

$$\partial\mathbf{B}/\partial t = \nabla \wedge (\mathbf{u} \wedge \mathbf{B}). \tag{25}$$

Hence

$$\frac{D}{Dt}\left(\frac{\mathbf{A}\,.\,\mathbf{B}}{\rho}\right) = \frac{\mathbf{B}}{\rho}\,.\,\nabla(\mathbf{A}\,.\,\mathbf{u} + \phi), \tag{26}$$

which may be compared with (14). It follows that

$$\int \mathbf{A}\,.\,\mathbf{B}\,dV = \text{const.} \tag{27}$$

provided $\mathbf{B}\,.\,\mathbf{n} = 0$ on the surface S of V. This result was proved (under slightly more restrictive conditions) by Woltjer (1958*a*). The interpretation of the invariant in terms of conservation of knottedness of magnetic lines of force (which are frozen in the fluid) is immediate. Note that the value of

$$\int_V \mathbf{A}\,.\,\mathbf{B}\,dV$$

† The term 'blob' of vorticity will be used to indicate a vorticity distribution $\boldsymbol{\omega}(\mathbf{x})$ that is entirely confined within some closed surface S of finite extent, i.e. $\boldsymbol{\omega} \equiv 0$ outside S.

is independent of the choice of gauge of $\mathbf{A}$; it is determined uniquely by the field $\mathbf{B}$ and the volume V.

The equation of motion is now

$$\rho(D\mathbf{u}/Dt) = -\nabla p + \mathbf{j} \wedge \mathbf{B}, \tag{28}$$

where $\mathbf{j} = \nabla \wedge \mathbf{B}$. From (28) and the induction equation in the form

$$\frac{D}{Dt}\left(\frac{\mathbf{B}}{\rho}\right) = \frac{\mathbf{B}}{\rho}.\nabla\mathbf{u}, \tag{29}$$

we may deduce (as for the case of vorticity) that

$$\frac{D}{Dt}\left(\frac{\mathbf{B}.\mathbf{u}}{\rho}\right) = \frac{\mathbf{B}}{\rho}.\nabla(-h+\tfrac{1}{2}q^2). \tag{30}$$

Hence

$$\int_V \mathbf{u}.\mathbf{B}\,dV = \text{const.} \tag{31}$$

provided again that $\mathbf{B}.\mathbf{n} = 0$ on S (cf. Woltjer 1958*b*).

It is evident, from (7) and (8), and similar formulae for $\mathbf{A}$ in terms of $\mathbf{B}$, that

$$\int \mathbf{u}.\mathbf{B}\,dV = -\frac{1}{4\pi}\iint \frac{\mathbf{R}.\boldsymbol{\omega}(\mathbf{x}') \wedge \mathbf{B}(\mathbf{x})}{R^3}\,dV = \int \mathbf{A}.\boldsymbol{\omega}\,dV. \tag{32}$$

The integrals are determined by the fields $\boldsymbol{\omega}$ and $\mathbf{B}$ (and of course by the volume of integration V); in order to emphasize this fact, it may be useful to introduce the notation

$$F\{\boldsymbol{\omega}, \mathbf{B}\} = \int_V \mathbf{u}.\mathbf{B}\,dV = F\{\mathbf{B}, \boldsymbol{\omega}\}. \tag{33}$$

Then also,

$$\int_V \mathbf{A}.\mathbf{B}\,dV = F\{\mathbf{B}, \mathbf{B}\} = I\{\mathbf{B}\} \quad \text{say}. \tag{34}$$

If $\mathbf{B} = 0$ except in flux filament $C_1', C_2', \ldots, C_m'$ with strengths $\Phi_1, \ldots, \Phi_m$, then

$$F\{\boldsymbol{\omega}, \mathbf{B}\} = \sum_i \Phi_i \oint_{C_i'} \mathbf{u}.d\mathbf{l} = \sum_i \Phi_i K_i', \tag{35}$$

where K_i' is the flux of vorticity through C_i'. This quantity is constant, because, although Kelvin's theorem does not now hold for an arbitrary curve (the Lorentz force $\mathbf{j} \wedge \mathbf{B}$ being, in general, rotational) it *does* hold if C is a closed $\mathbf{B}$-line; for then

$$\oint_C \mathbf{j} \wedge \mathbf{B}.d\mathbf{l} = 0,$$

(Shercliff 1965, problem 4·7). Hence again the integral $F\{\boldsymbol{\omega}, \mathbf{B}\}$ may be interpreted as a measure of the degree of mutual knottedness of the two fields $\boldsymbol{\omega}$ and $\mathbf{B}$; this remains constant even though the vortex lines are no longer frozen in the fluid.

6. Some examples of flows for which $I \neq 0$

The only situations considered so far which definitely give a non-zero value for I are those of § 1 in which discrete vortex filaments are linked or knotted. Such a configuration may seem physically artificial and unlikely and one might be tempted to conclude that flows with continuous vorticity and with $I \neq 0$ are unlikely to occur naturally. Such a conclusion would not however be justified. The influence of viscosity near solid boundaries in causing changes in I has already been mentioned. A blob of vorticity may be generated by the sudden acceleration of part of a solid boundary surrounding a fluid. If, say, a right-handed impulsive wrench† is applied to an immersed body, then it is more than likely that some of the helicity imparted to the body will be transferred to the fluid via the vorticity shed from the boundary during the initial stages of the motion. The vortex lines then become linked during the shedding process and remain linked thereafter. The spiral trailing vortex system behind an advancing propeller provides perhaps the best example. Any advancing rotating body must likewise leave a helical vorticity distribution in its wake.

Some examples of flows with particular symmetries will help to clarify the character of the linkages that are likely to occur in situations of practical interest.

(a) Two-dimensional incompressible flow

For a velocity field of the form

$$\mathbf{u} = \left(\frac{\partial \psi}{\partial y}, -\frac{\partial \psi}{\partial x}, w\right), \tag{36}$$

where $\psi = \psi(x, y, t)$, $w = w(x, y, t)$, we have

$$\mathbf{u} \cdot \boldsymbol{\omega} = \nabla\psi \cdot \nabla w - w\nabla^2\psi. \tag{37}$$

Provided the flow is localized in the (x, y)-plane, ($|\boldsymbol{\omega}| = 0(r^{-3})$ as $r^2 = x^2 + y^2 \to \infty$ is certainly a sufficient restriction), we may take V to be the volume between any two planes $z = \text{const.}$ at unit distance apart, (the contribution to the surface integral (17) from these two planes then cancelling), and the invariant I degenerates to an integral over the (x, y)-plane,

$$I = \iint (\nabla\psi \cdot \nabla w - w\nabla^2\psi)\, dx\, dy = -2\iint w\nabla^2\psi\, dx\, dy. \tag{38}$$

The conditions for steady flow are

$$w = C(\psi), \quad p/\rho + \tfrac{1}{2}q^2 = H(\psi), \tag{39}$$

and

$$\nabla^2\psi = \frac{dH}{d\psi} - C\frac{dC}{d\psi} = f(\psi) \quad \text{say.} \tag{40}$$

In this case,

$$I = -2\iint f(\psi)\, C(\psi)\, dx\, dy. \tag{41}$$

† I.e. an impulsive force $\mathbf{F}$ and couple $\mathbf{G}$ with $\mathbf{F} \cdot \mathbf{G} > 0$.

The simplest explicit example is perhaps that of the rectilinear vortex with an axial motion confined to the vortex:

$$f(\psi) = \begin{cases} -\omega_0 & (r < a), \\ 0 & (r > a), \end{cases} \tag{42}$$

where $r = (x^2+y^2)^{\frac{1}{2}}$. In this case, $I = 2\omega_0 Q,$ (43)

where
$$Q = \iint w(x,y)\,dx\,dy. \tag{44}$$

(It is assumed that Q is finite; the manner in which the vortex lines are closed at $r = \infty$ is then immaterial.) Thus $I \gtrless 0$ according as ω_0 and Q have the same or opposite signs, that is, according as the sense of the net screw of the vortex is right-handed or left-handed.

The vortex lines of steady flows of this type are helices which spiral round on the cylindrical surfaces $H =$ const., or equivalently $\psi =$ const. There must exist at least one point (x_0, y_0) at which $\nabla\psi = 0$, and the line $x = x_0$, $y = y_0$ is itself a vortex line, lying on a degenerate member of the family of surfaces $\psi =$ const. This line may be termed a 'vortex axis'; all vortex lines in the neighbourhood of a vortex axis spiral round it.

(*b*) *Axisymmetric incompressible flow*

Suppose now that, relative to cylindrical polar co-ordinates (x, r, ϕ), the velocity has components

$$\mathbf{u} = \left(\frac{1}{r}\frac{\partial\psi}{\partial r},\ -\frac{1}{r}\frac{\partial\psi}{\partial x},\ w\right), \tag{45}$$

where
$$\psi = \psi(x,r,t), \quad w = w(x,r,t).$$

Then
$$\mathbf{u}.\boldsymbol{\omega} = r^{-2}[\nabla\psi.\nabla(rw) - rwD^2\psi], \tag{46}$$

where
$$D^2 \equiv \frac{\partial^2}{\partial x^2} + r\frac{\partial}{\partial r}\frac{1}{r}\frac{\partial}{\partial r}, \tag{47}$$

and for an axisymmetric blob of vorticity,

$$I = 2\pi\iint\left[\frac{1}{r}\nabla\psi.\nabla(rw) - wD^2\psi\right]dx\,dr. \tag{48}$$

Provided $w|\nabla\psi|$ is everywhere finite, and $o(R^{-2})$ as $R = (r^2+x^2)^{\frac{1}{2}} \to \infty$, this may be transformed by means of the divergence theorem to

$$I = -4\pi\iint wD^2\psi\,dx\,dr. \tag{49}$$

Hence the integrated product of the swirl w and the azimuthal vorticity $-r^{-1}D^2\psi$ is invariant in any axisymmetric unsteady inviscid flow. The result may be regarded as a particular consequence of the fact that for any material toroidal filament, wr and $r^{-1}\omega_\phi = r^{-2}D^2\psi$ are invariant; integration of the product $r^{-1}wD^2\psi$ over the volume of fluid, with $dV = 2\pi r\,dx\,dr$ then gives the integral (49).

The conditions for *steady* flow in this case (Batchelor 1967, §7.5) are

$$rw = C(\psi), \quad p/\rho + \tfrac{1}{2}q^2 = H(\psi), \tag{50}$$

and
$$D^2\psi = r^2\frac{dH}{d\psi} - C\frac{dC}{d\psi}. \tag{51}$$

An interesting family of solutions, each of which represents a blob of vorticity confined to the sphere $R < a$, exists when $C(\psi)$ and $H(\psi)$ have the simple forms
$$H = H_0 + \lambda\psi, \quad C = \pm\alpha\psi, \tag{52}$$
where H_0, λ and α are constants. In this case, (51) becomes
$$D^2\psi = \lambda r^2 - \alpha^2\psi, \tag{53}$$
and this admits the solution in spherical polars (R, θ, ϕ),
$$\psi = R^2 \sin^2\theta \left[\frac{\lambda}{\alpha^2} + A\left(\frac{a}{R}\right)^{\frac{3}{2}} J_{\frac{3}{2}}(\alpha R)\right], \tag{54}$$
where A is a constant. There are other solutions of (53) with more complicated dependence on θ, but the interest of solutions of the form (54) is that they can be matched, by suitable choice of the constants λ, α and A to an irrotational stream, represented by
$$\psi = \tfrac{1}{2}U\{R^2 - (a^3/R)\}\sin^2\theta, \tag{55}$$
for $R > a$. We have to satisfy
$$\psi = 0, \quad \frac{\partial\psi}{\partial R} = \tfrac{3}{2}Ua\sin^2\theta, \quad \text{on} \quad R = a. \tag{56}$$

These ensure that the surface $R = a$ is a stream-surface, and that the velocity is continuous across it; continuity of pressure, given by (50), can then also be satisfied. These conditions give respectively
$$\frac{\lambda}{\alpha^2} = -AJ_{\frac{3}{2}}(\alpha a), \quad U = -\tfrac{2}{3}AJ_{\frac{5}{2}}(\alpha a), \tag{57}$$
and a doubly infinite family of solutions is obtained by varying the parameters A and αa. U is the speed at which the vortex propagates relative to the fluid at infinity.

Two possibilities deserve particular comment. If $J_{\frac{3}{2}}(\alpha a) = 0$, then $\lambda = 0$, and (52) and (53) together imply that $\boldsymbol{\omega} = \pm\alpha\mathbf{u}$; the resulting velocity field is then exactly analogous to the 'force-free' magnetic field obtained (among others) by Chandrasekhar† (1956). Secondly, if $J_{\frac{5}{2}}(\alpha a) = 0$, then $U = 0$, and so the fluid is at rest outside the sphere $R = a$; this is exactly analogous to the magnetostatic solution proposed by Prendergast (1957) as a model for the equilibrium structure of a magnetic star (and described by Roberts 1967, §4.7).

The total helicity of the vortex described by (52), (54) is given by (49), i.e.
$$I = -4\pi\iint \frac{\pm\alpha\psi}{r}(\lambda r^2 - \alpha^2\psi)\,dx\,dr.$$

† The governing equations in the magnetostatic problem are $\mathbf{j} \wedge \mathbf{B} = \nabla p$, $\mathbf{j} = \nabla \wedge \mathbf{B}$, $\nabla . \mathbf{B} = 0$, and the analogy with the situation under consideration here is that between the variables
$$\mathbf{u} \leftrightarrow \mathbf{B}, \quad \boldsymbol{\omega} \leftrightarrow \mathbf{j}, \quad H \leftrightarrow p.$$

After straightforward manipulation, this reduces to

$$I = \pm \tfrac{16}{3}\pi a^2 A^2 f(\alpha a), \tag{58}$$

where
$$f(z) = \tfrac{1}{2}z^2\{z(J_{\frac{3}{2}}(z))^2 - zJ_{\frac{1}{2}}(z)\, J_{\frac{5}{2}}(z) - 2J_{\frac{3}{2}}(z)\, J_{\frac{1}{2}}(z)\}. \tag{59}$$

The function $f(z)$ has the asymptotic behaviour

$$f(z) \sim \left\{ \begin{array}{ll} \dfrac{\pi z^6}{1560} & \text{as} \quad z \to 0, \\ \dfrac{\pi z^2}{4} & \text{as} \quad z \to \infty. \end{array} \right\} \tag{60}$$

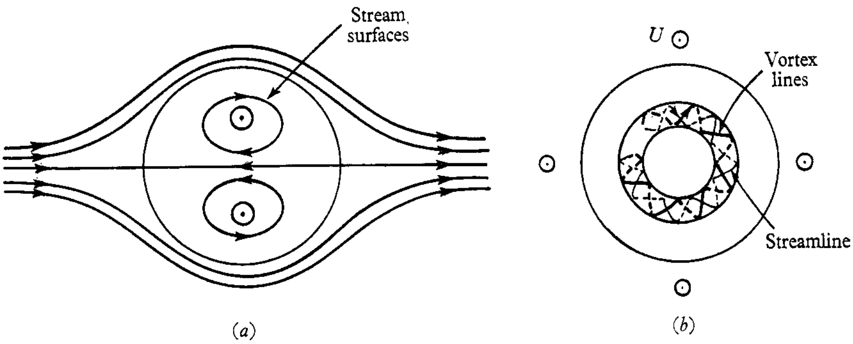

FIGURE 3. Stream-surfaces, streamlines and vortex lines for the spherical vortices described by (54), (52). In (b) the vortex is viewed from the direction of the stream at infinity.

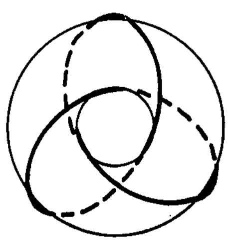

FIGURE 4. One of the knotted vortex lines of the spherical vortex represented by the stream function (54) and the circulation (52).

The choice of sign in (58) corresponds to the choice in (52); both 'right-handed' and 'left-handed' vortices are possible.

The surfaces ψ = constant for $R < a$ consist in all cases of a family of nested tori, the sphere $R = a$ itself being the limiting outer member of the family. The innermost member of each family of tori degenerates into a circle which is both a streamline and a vortex line: it is located on the plane $\theta = \frac{1}{2}\pi$ at a point where $\partial\psi/\partial R = 0$, i.e. $\psi = \psi_{\max}$, say, (at least one such point exists). The surfaces ψ = const. are sketched in figure 3(a) for the simplest case in which there is only one such 'vortex axis' within the sphere. The streamlines and the vortex lines lie on these surfaces, as indicated in figure 3(b), which is a view of the eddy from along its axis of symmetry.

If any one vortex line is followed in the direction of increasing ϕ the value of z on that line varies periodically; the *pitch* p of the vortex line may conveniently be defined as twice the increase in ϕ, between successive zeros of z. This quantity clearly increases continuously from zero to infinity as ψ increases from zero (on $R = a$) to $\psi_{\max}$ (on the vortex axis). If $p = 2\pi m/n$ where m and n are integers prime to each other, then the vortex line will close on itself after traversing the smaller circumference of the torus n times and the larger circumference m times. Such vortex lines are self-knotted if $m \geqslant 2$, $n \geqslant 3$; the corresponding knot is known as the torus knot of type m, n. For example, if $m = 2$ and $n = 3$, so that $p = \frac{4}{3}\pi$, the vortex line is in the form of the trefoil knot, as indicated in figure 4. It is interesting that *every* torus knot is represented once and only once among all the vortex lines of each member of the family of flows represented by the stream function (54), together with the circulation (52).

It is a pleasure to acknowledge the stimulus of several discussions with Professor G. K. Batchelor on the topic of this paper. It was he in particular who recognized the physical significance of Woltjer's second invariant, equation (31). I am also indebted to Dr K. J. Whiteman who drew my attention to the family of torus knots referred to in the last paragraph.

REFERENCES

Batchelor, G. K. 1967 *An Introduction to Fluid Dynamics*. Cambridge University Press.

Chandrasekhar, S. 1956 *Proc. Nat. Acad. Sci. U.S.A.* **42**, 1–5.

Crowell, R. H. & Fox, R. H. 1964 *Introduction to Knot Theory*. Ginn.

Gauss 1833 Werke. Königlichen Gesellschaft der Wissenschaften zu Göttingen, 1877, **5**, 605.

Prendergast, K. 1957 *Astrophys. J.* **123**, 498.

Roberts, P. H. 1967 *An Introduction to Magnetohydrodynamics*. Longmans.

Shercliff, J. A. 1965 *A Textbook of Magnetohydrodynamics*. Oxford: Pergamon.

Tait, P. G. 1898 *Scientific Papers I*. Cambridge University Press.

Thomson, W. (Lord Kelvin) 1868 *Trans. Roy. Soc. Edin.* **25**, 217–260.

Woltjer, L. 1958*a* *Proc. Nat. Acad. Sci. U.S.A.* **44**, 489–491.

Woltjer, L. 1958*b* *Proc. Nat. Acad. Sci. U.S.A.* **44**, 833–841.