Dynamics of the Atmosphere and the Ocean

Lecture 4

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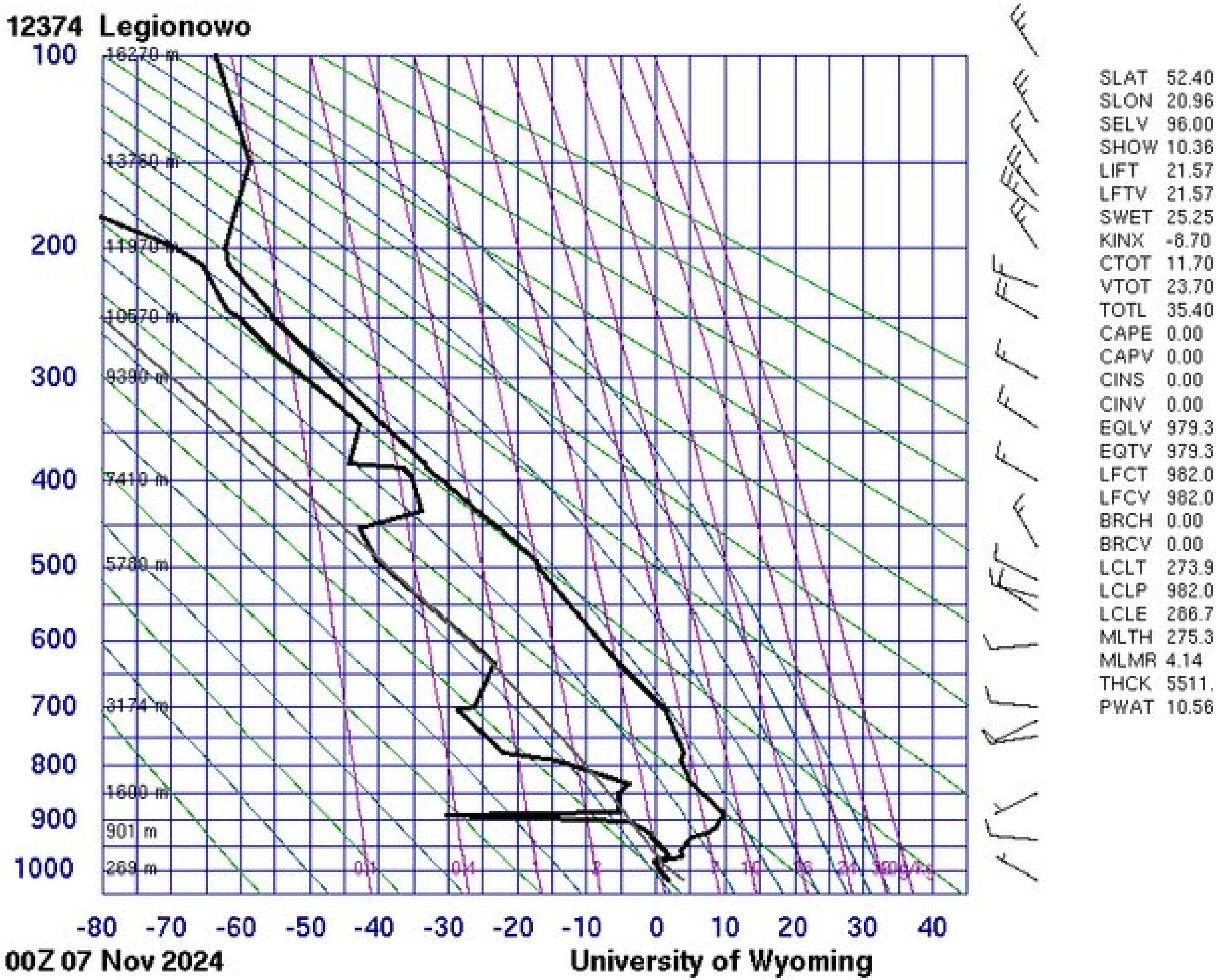
2025-2026 Fall



Pressure coordinates

Let's consider primitive equations **for the atmosphere** approximated by an ideal gas:

$$\frac{D\boldsymbol{u}}{Dt} + \boldsymbol{f} \times \boldsymbol{u} = -\frac{1}{\rho}\nabla p,$$

$$\frac{\partial p}{\partial z} = -\rho g,$$

$$\frac{D\theta}{Dt} = 0,$$

$$\frac{D\rho}{Dt} + \rho\nabla\cdot\boldsymbol{v} = 0,$$

Here $p=\rho RT$ and $\theta=T(p_R/p)^{R/cp}$ and p_R is the reference pressure (usually 1000hPa). These equations can be transformed from Cartesian (x,y,z) to pressure (x,y,p) coordinates. The analog to the vertical velocity is: $\omega=Dp/Dt$ and the advective derivative has the form:

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + \boldsymbol{u}\cdot\nabla_p + \omega\frac{\partial}{\partial p}.$$

The horizontal and time derivatives are taken at constant pressure. However, x and y are still purely horizontal coordinates, perpendicular to the vertical (z) axis. The operator D=Dt is the same in pressure or height coordinates because is y the total derivative of some property of a fluid parcel. However, the individual terms comprising it in general differ between height and pressure coordinates.

To obtain an expression for the pressure force, first consider a general vertical coordinate:

$$\left(\frac{\partial}{\partial x}\right)_\xi = \left(\frac{\partial}{\partial x}\right)_z + \left(\frac{\partial z}{\partial x}\right)_\xi \frac{\partial}{\partial z}.$$

The above for $\xi = p$ gives:

$$0 = \left(\frac{\partial p}{\partial x}\right)_z + \left(\frac{\partial z}{\partial x}\right)_p \frac{\partial p}{\partial z},$$

Applying hydrostatic relationship:

$$\left(\frac{\partial p}{\partial x}\right)_z = \rho \left(\frac{\partial \Phi}{\partial x}\right)_p ,$$

where $\Phi = gz$ is geopotential. Finally,

$$\frac{1}{\rho}\nabla_z p = \nabla_p \Phi,$$

$$\frac{\partial \Phi}{\partial p} = -\alpha.$$

Mass continuity in pressure coordinates takes the form:

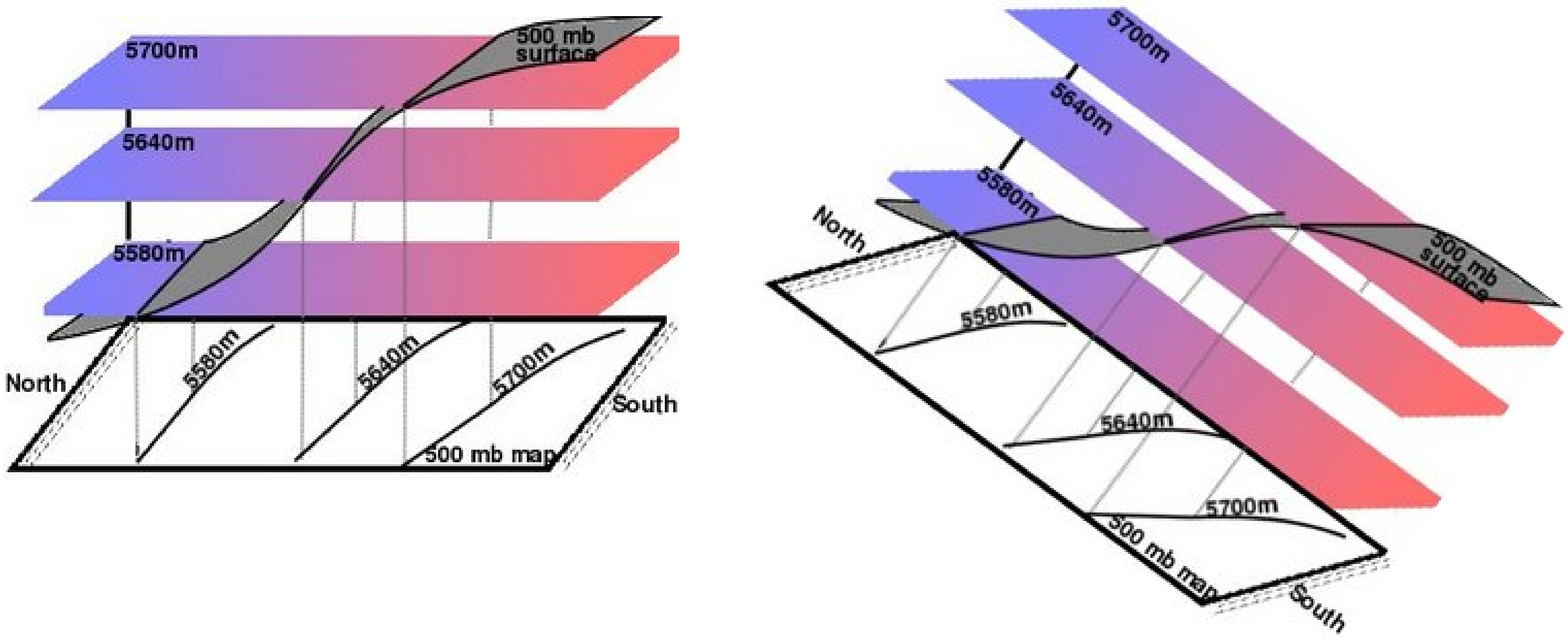
$$\nabla_p \cdot \boldsymbol{u} + \frac{\partial \omega}{\partial p} = 0,$$

And the whole set of primitive equations can be written as:

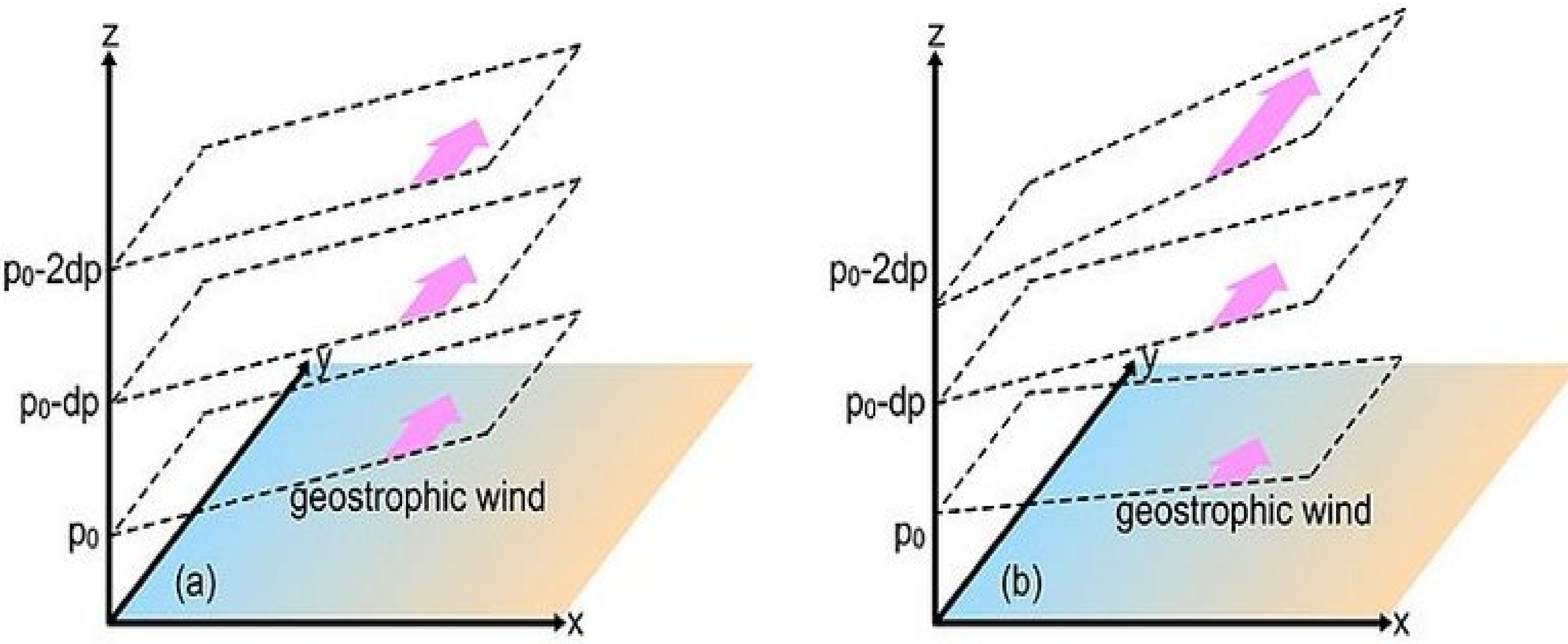
$$\frac{\mathrm{D}\boldsymbol{u}}{\mathrm{D}t} + \boldsymbol{f} \times \boldsymbol{u} = -\nabla_p \Phi$$
$$\frac{\partial \Phi}{\partial p} = -\alpha$$
$$\frac{\mathrm{D}\theta}{\mathrm{D}t} = 0$$
$$\nabla_p \cdot \boldsymbol{u} + \frac{\partial \omega}{\partial p} = 0$$

Together with the ideal gas equation and potential temperature definition.

These are **not quite isomorphic to the Boussinesq equations**, because the hydrostatic equation is: $D\Phi/Dp=-\alpha=-(\theta R/p_R)(p_R/p)^{1/\gamma}$ and not, as we would require, $D\Phi/Dp=-\theta$.



Schematic difference between Cartesian coordinates (left) and pressure coordinates (right).



Notice, that horizontal temperature gradients result in changes in the inclination constant pressure surfaces. Such a situation is called “baroclinicity” (right).

Baroclinicity. Thermal wind.

You might notice from presented potential fields that distances between isobaric surfaces may differ. What is s the mechanism of these differences?

Consider horizontal flow in geostrophic balance in Boussinesq or anelastic notation:

$$-f v_g = -\frac{\partial \phi}{\partial x} = -\frac{1}{a \cos \vartheta} \frac{\partial \phi}{\partial \lambda}$$
$$f u_g = -\frac{\partial \phi}{\partial y} = -\frac{1}{a} \frac{\partial \phi}{\partial \vartheta}$$

Consider change of this balance with height, accounting for $\partial \Phi / \partial z = b$ which gives:

$$-f \frac{\partial v_g}{\partial z} = -\frac{\partial b}{\partial x} = -\frac{1}{a \cos \lambda} \frac{\partial b}{\partial \lambda}$$
$$f \frac{\partial u_g}{\partial z} = -\frac{\partial b}{\partial y} = -\frac{1}{a} \frac{\partial b}{\partial \vartheta}$$

The above is known as “thermal wind balance”. Notice that b relates to horizontal temperature gradients in the atmosphere and density gradients in the ocean.

As you see with the previous slide one of the difficulties with pressure coordinates is the lower boundary condition. Using:

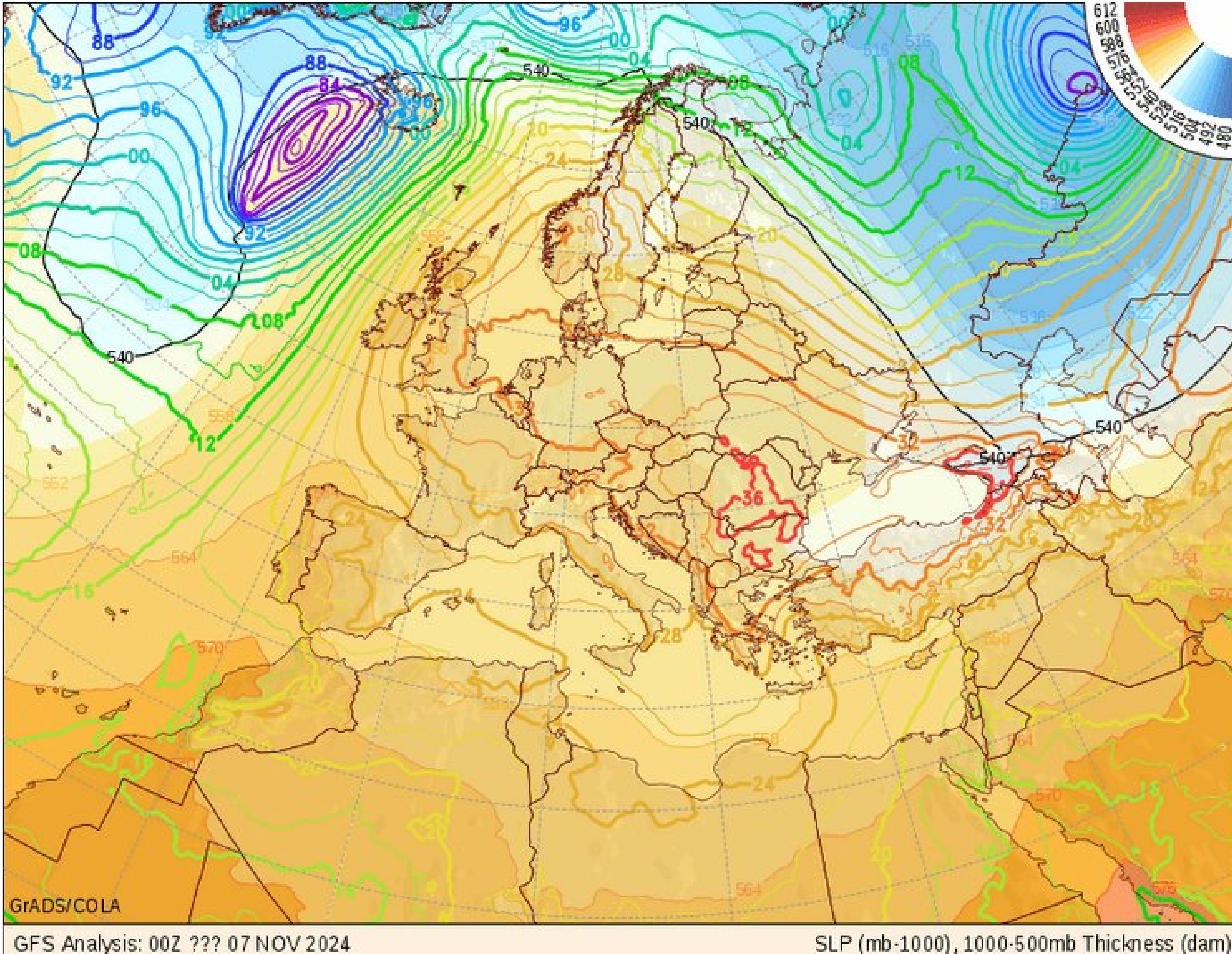
$$w \equiv \frac{Dz}{Dt} = \frac{\partial z}{\partial t} + \boldsymbol{u} \cdot \nabla_p z + \omega \frac{\partial z}{\partial p},$$

and hydrostatic equation , the boundary condition of $\omega=0$ at $z=z_s$ becomes

$$\frac{\partial \Phi}{\partial t} + \boldsymbol{u} \cdot \nabla_p \Phi - \alpha \omega = 0$$

In theoretical studies one may assume $\omega=0$ at $p(x,y,z_s,t)$. In practice fact that the lower boundary is not a coordinate surface has to be accounted for. Additionally for uneven (topography) lower boundary so-called sigma coordinates are often used.

Sigma coordinates may use height itself as a measure of displacement (typical in oceanic applications) or use pressure (typical in atmospheric applications $\sigma=p/p_s$ where $p_s(x,y,z_s,t)$ is the surface pressure.
The difficulty of applying the above is replaced by a prognostic equation for the surface.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SLP (mb-1000), 1000-500mb Thickness (dam)

In pressure coordinates thermal wind balance can be obtained e.g. taking geostrophic balance in form:

$$\boldsymbol{f} \times \boldsymbol{u}_g = -\nabla_p \Phi$$

and looking for its change with pressure, remembering that $D\Phi/Dp = -\alpha$:

$$\boldsymbol{f} \times \frac{\partial \boldsymbol{u}_g}{\partial p} = \nabla_p \alpha = \frac{R}{p} \nabla_p T,$$

Where we accounted for ideal gas equation $p\alpha = RT$. In component form the above is:

$$-f \frac{\partial v_g}{\partial p} = \frac{R}{p} \frac{\partial T}{\partial x}, \qquad f \frac{\partial u_g}{\partial p} = \frac{R}{p} \frac{\partial T}{\partial y}.$$

Here temperature horizontal gradients are clearly seen.

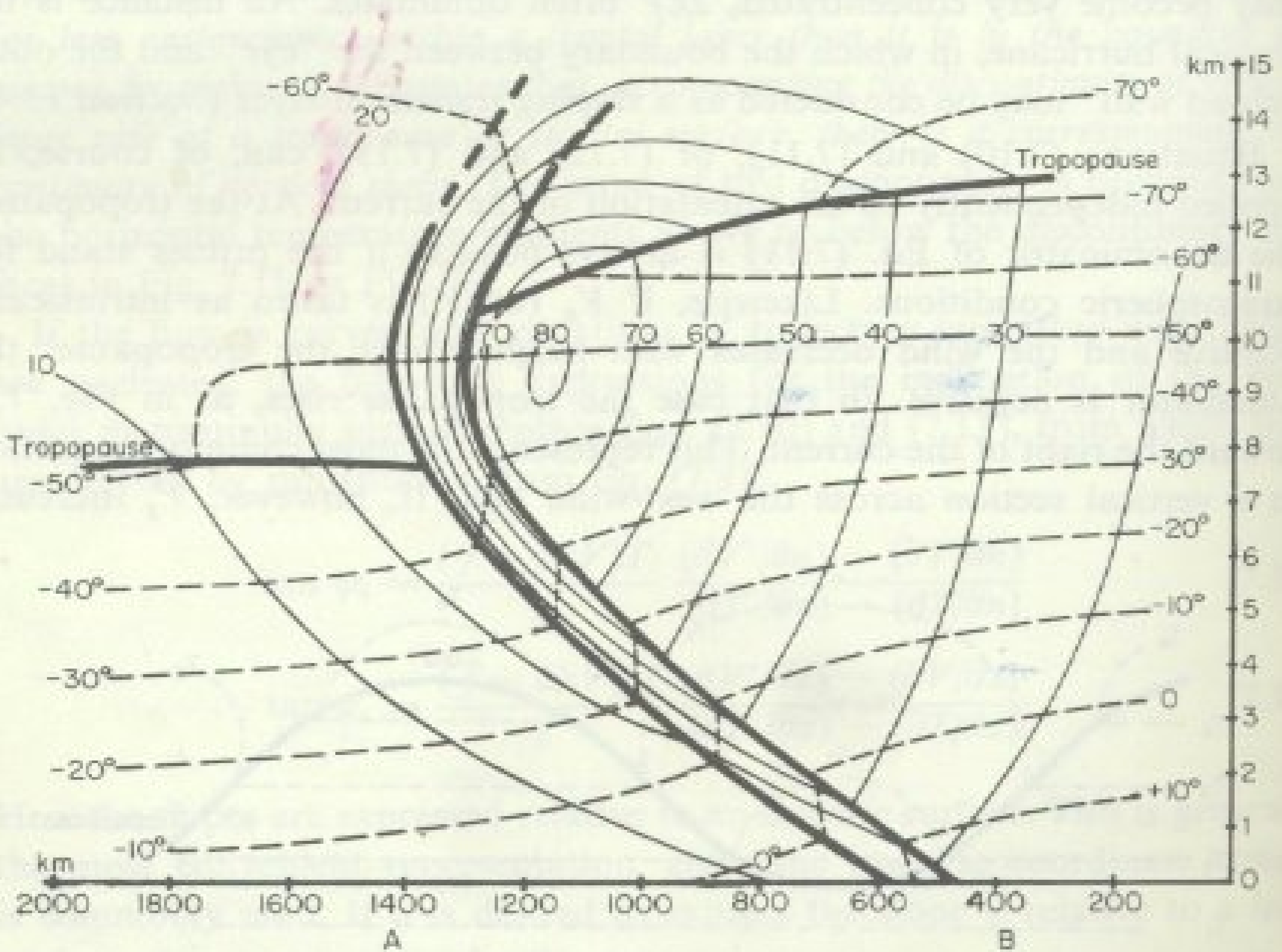
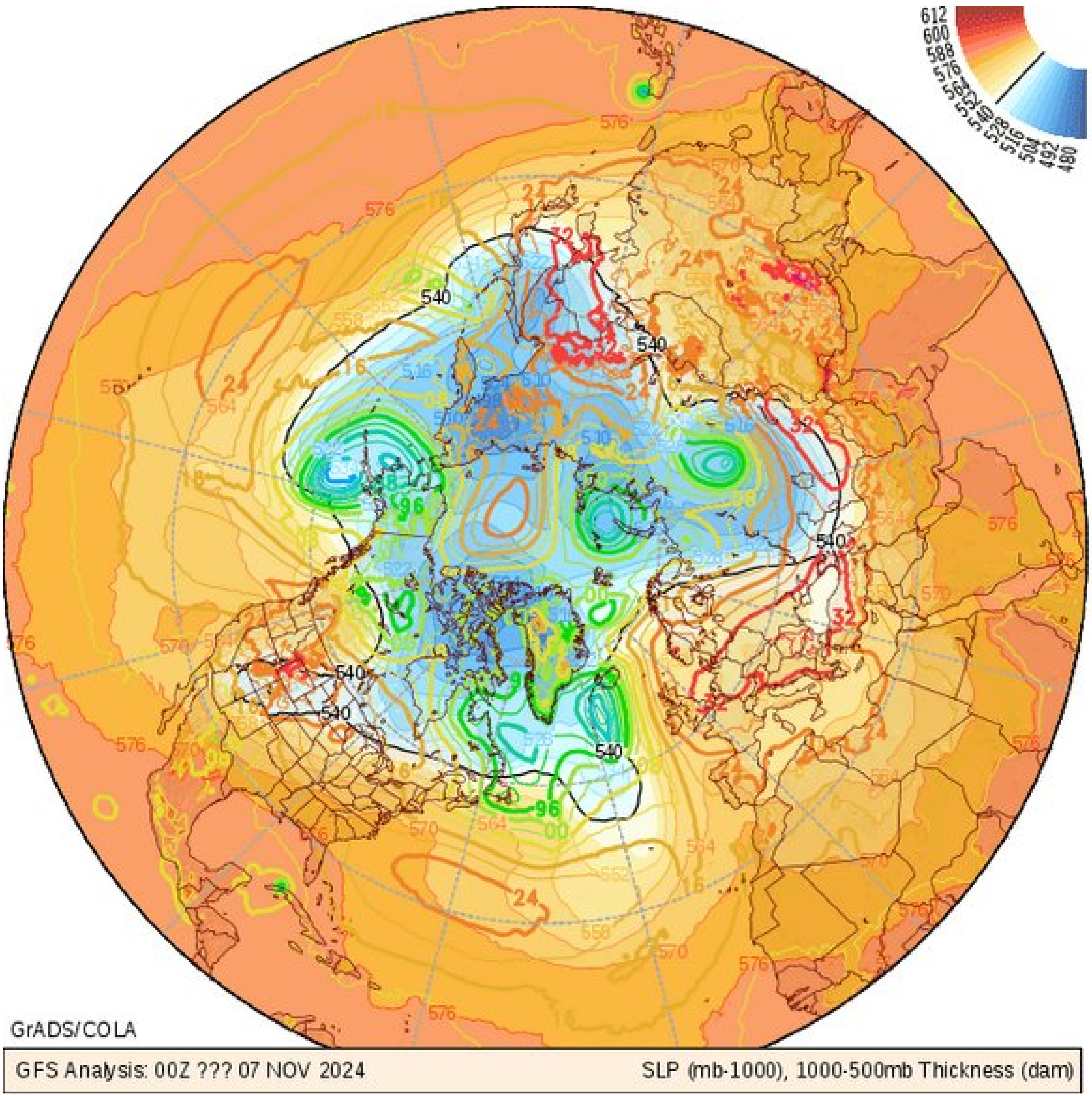


FIG. 7.4 Schematic isotherms (dashed lines, °C) and isotachs (thin solid lines, meters per second) in the polar front zone. Heavy lines are tropopauses and boundaries of frontal layer. (Adapted from analysis model by Berggren, 1952.)

Thermal wind and polar front.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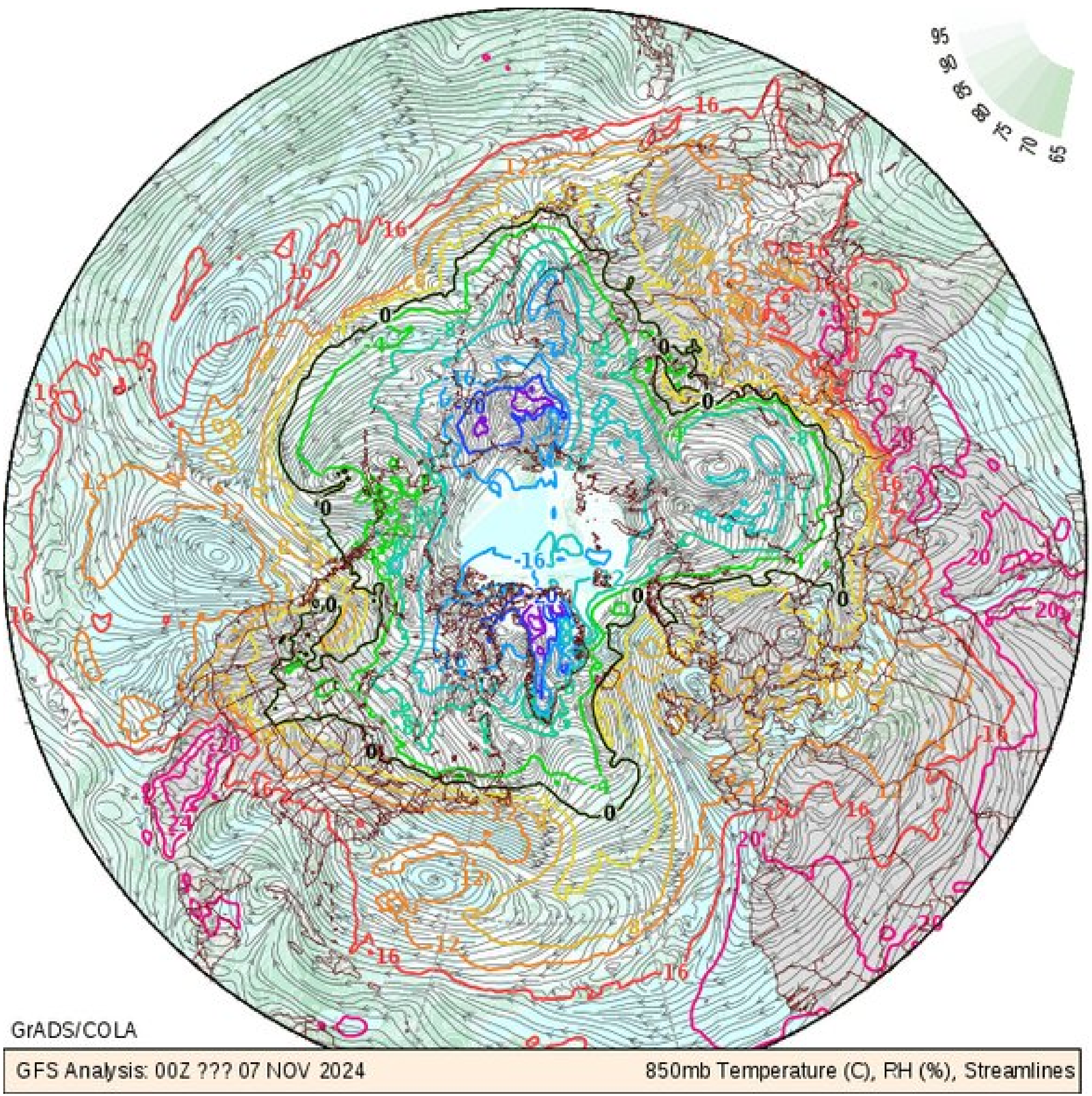
SLP (mb-1000), 1000-500mb Thickness (dam)


95
90
85
80
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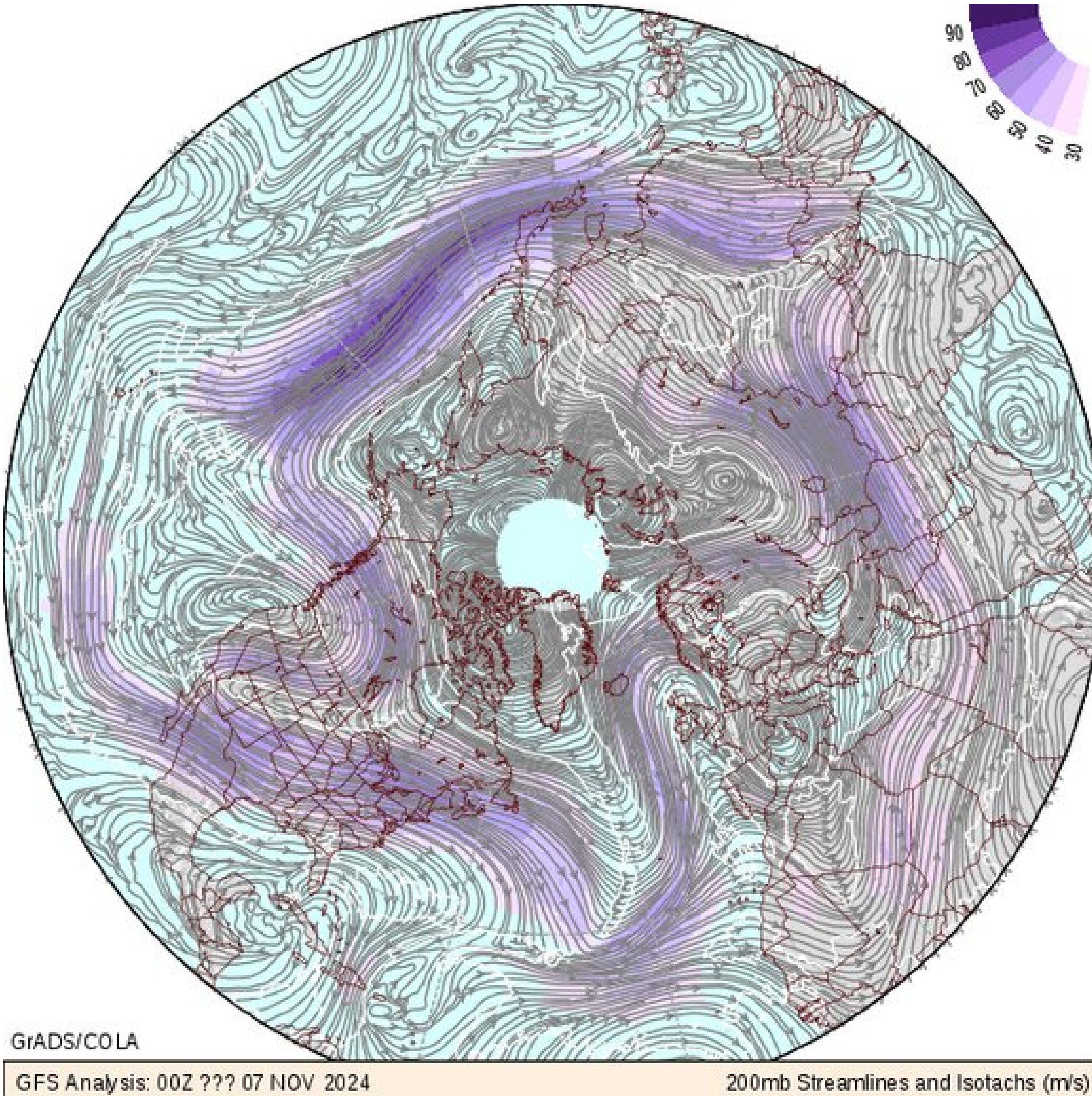
850mb Temperature (C), RH (%), Streamlines


90
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GFS Analysis: 00Z ??? 07 NOV 2024
200mb Streamlines and Isotachs (m/s)

Vertical temperature profile in adiabatic isobaric atmosphere

Consider adiabatic atmosphere, air as an ideal gas:

$$c_v\, dT + p dv = 0, \quad pv = RT\,.$$

Then:

$$c_v\, d(\ln T) + R\, d(\ln v) = 0, \quad T^{c_v} v^R = const.$$

Let's introduce:

$$\gamma = {c_p}/{c_v} = {(c_v + R)}/{c_v} = 1 + \frac{R}{c_v} = 1 + \frac{R}{\frac{5}{2}R} = 1 + \frac{2}{5} = 1.4$$

$$\kappa = {R}/{c_p} = {(\gamma - 1)}/{\gamma} \cong 0.286$$

which results in:

$$Tv^{\gamma-1} = const$$

$$Tp^{-\kappa} = const$$

$$pv^{\gamma} = const$$

$$Tp^{-\kappa} = const$$

Let's define temperature conserved in the course of adiabatic changes of pressure:

$$\theta p_0^{-\kappa} = Tp^{-\kappa}$$

$$\frac{\theta}{T} = \left(\frac{p_0}{p}\right)^{\kappa}$$

Here p_0 is the reference pressure, conveniently taken as 1000hPa.

Using hydrostatic relation and first thermodynamic principle:

$$c_p dT - v dp = 0$$

$$dp = -\rho g dz$$

$$dp = -\frac{g}{v} dz \quad \Rightarrow v dp = -g dz$$

one obtains: $$c_p dT + g dz = 0$$

which can be expressed as: $$\Gamma_d \equiv -\frac{dT}{dz} = \frac{g}{c_p}$$ $$\Gamma_d = \frac{g}{c_p} = \frac{9{,}81\,{}^{m}/_{s^2}}{1004\,{}^{J}/_{kg\cdot K}} \approx 9{,}8\,\frac{K}{km}$$

Γ_d is dry adiabatic lapse rate.

Potential temperature:

$$\theta p_0^{-\kappa} = T p^{-\kappa}$$

$$\frac{\theta}{T} = \left(\frac{p_0}{p}\right)^{\kappa}$$

Potential density:

$$p = \rho R T$$

$$p_0 = \rho_0 R \theta$$

$$\rho_o p_0^{\kappa-1} = \rho p^{\kappa-1}$$

STATIC INSTABILITY AND THE PARCEL METHOD

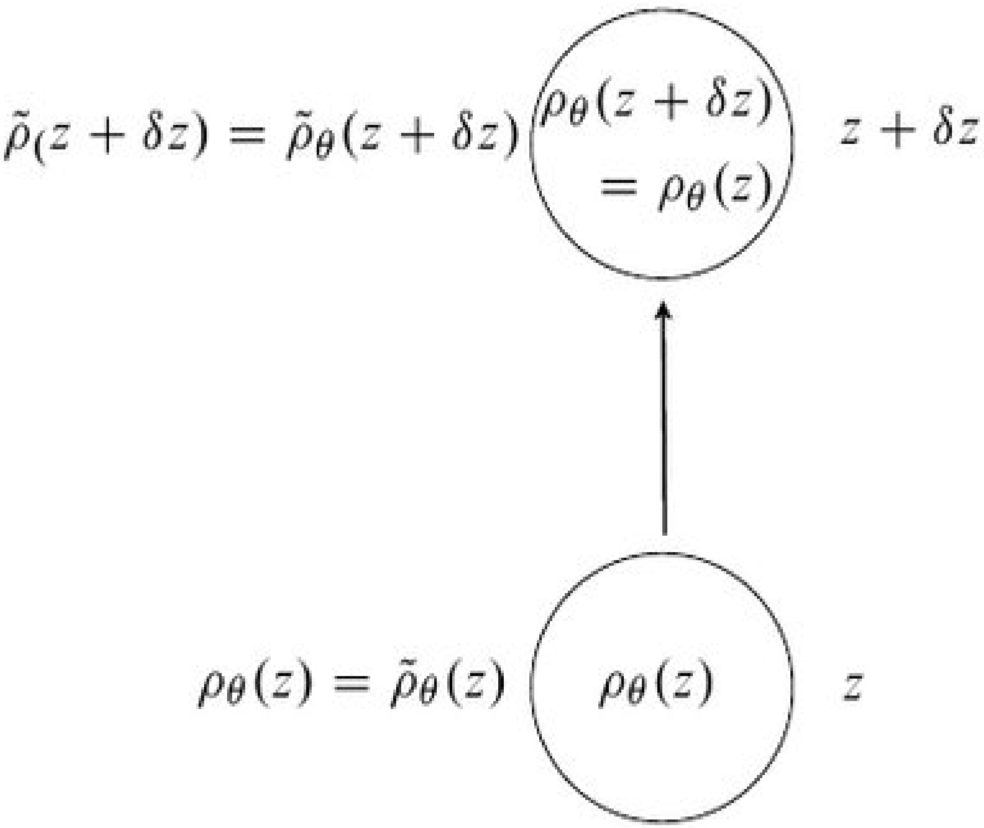


Figure 2.6 A parcel is adiabatically displaced upward from level z to $z+\delta z$. If the resulting density difference, $\delta\rho$, between the parcel and its new surroundings is positive the displacement is stable, and conversely. If $\widetilde{\rho}$ is the environmental values, and ρ_θ is potential density, we see that $\delta\rho = \widetilde{\rho}_\theta(z) - \widetilde{\rho}_\theta(z+\delta z)$

$$\delta\rho = \rho(z+\delta z) - \widetilde{\rho}(z+\delta z) = \widetilde{\rho}(z) - \widetilde{\rho}(z+\delta z) = -\frac{\partial\widetilde{\rho}}{\partial z}\delta z.$$

$$\begin{aligned}\delta\rho &= \rho(z+\delta z) - \widetilde{\rho}(z+\delta z) = \rho_\theta(z+\delta z) - \widetilde{\rho}_\theta(z+\delta z) \\ &= \rho_\theta(z) - \widetilde{\rho}_\theta(z+\delta z) = \widetilde{\rho}_\theta(z) - \widetilde{\rho}_\theta(z+\delta z),\end{aligned} \tag{2.218}$$

and therefore

$$\delta\rho = -\frac{\partial\widetilde{\rho}_\theta}{\partial z}\delta z. \tag{2.219}$$

where the right-hand side is the environmental gradient of potential density. If the right-hand-side is positive, the parcel is heavier than its surroundings and the displacement is stable. Thus, the conditiona for stability are:

$$\text{Stability}: \qquad \frac{\partial\widetilde{\rho}_\theta}{\partial z} < 0 \tag{2.220a}$$

$$\text{Instability}: \qquad \frac{\partial\widetilde{\rho}_\theta}{\partial z} > 0 \tag{2.220b}$$

The equation of motion of the fluid parcel is

$$\frac{\partial^2\delta z}{\partial t^2} = \frac{g}{\rho}\left(\frac{\partial\widetilde{\rho}_\theta}{\partial z}\right)\delta z = -N^2\delta z \tag{2.221}$$

where, noting that $\rho(z) = \widetilde{\rho}_\theta(z)$ to within $O(\delta z)$,

Brunt-Vaisala
frequency

$$\boxed{N^2 = -\frac{g}{\widetilde{\rho}_\theta}\left(\frac{\partial\widetilde{\rho}_\theta}{\partial z}\right)}. \tag{2.222}$$

This is a general expression for the buoyancy frequency, true in both liquids and gases. It is important to realize that the quantity $\widetilde{\rho}_\theta$ is the *locally-referenced* potential density

An ideal gas

In the atmosphere potential density is related to potential temperature by $\rho_\theta = p_R/(\theta R)$.

$$\boxed{N^2 = \frac{g}{\widetilde{\theta}}\left(\frac{\partial \widetilde{\theta}}{\partial z}\right)}, \tag{2.223}$$

where $\widetilde{\theta}$ refers to the environmental profile of potential temperature.

The negative of the rate of change of the temperature in the vertical is known as the *temperature lapse rate*, or often just the lapse rate, and the rate corresponding to $\partial\theta/\partial z = 0$ is called the *dry adiabatic lapse rate*. Using $\theta = T(p_0/p)^{R/c_p}$ and $\partial p/\partial z = -\rho g$ we find that the lapse rate and the potential temperature lapse rate are related by

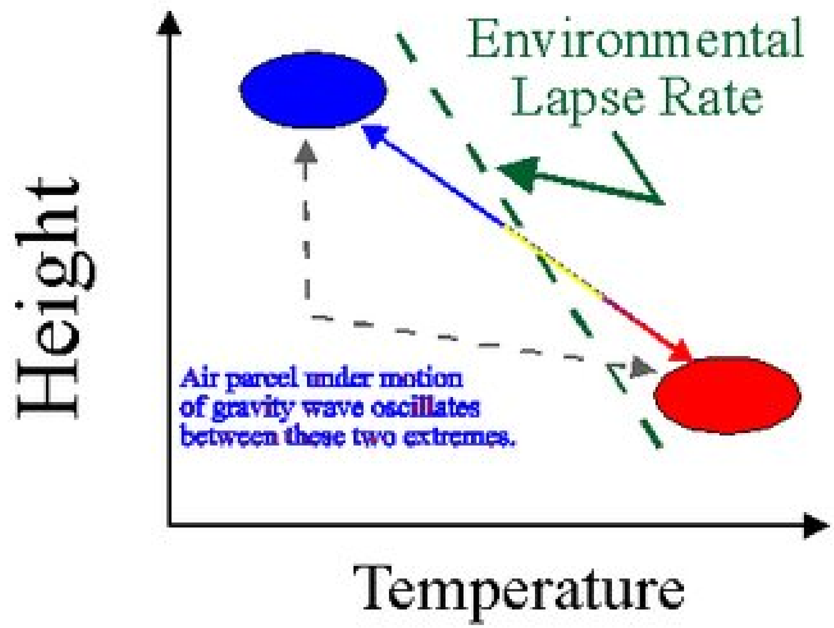
$$\frac{\partial \widetilde{T}}{\partial z} = \frac{\widetilde{T}}{\widetilde{\theta}}\frac{\partial \widetilde{\theta}}{\partial z} - \frac{g}{c_p}, \tag{2.228}$$

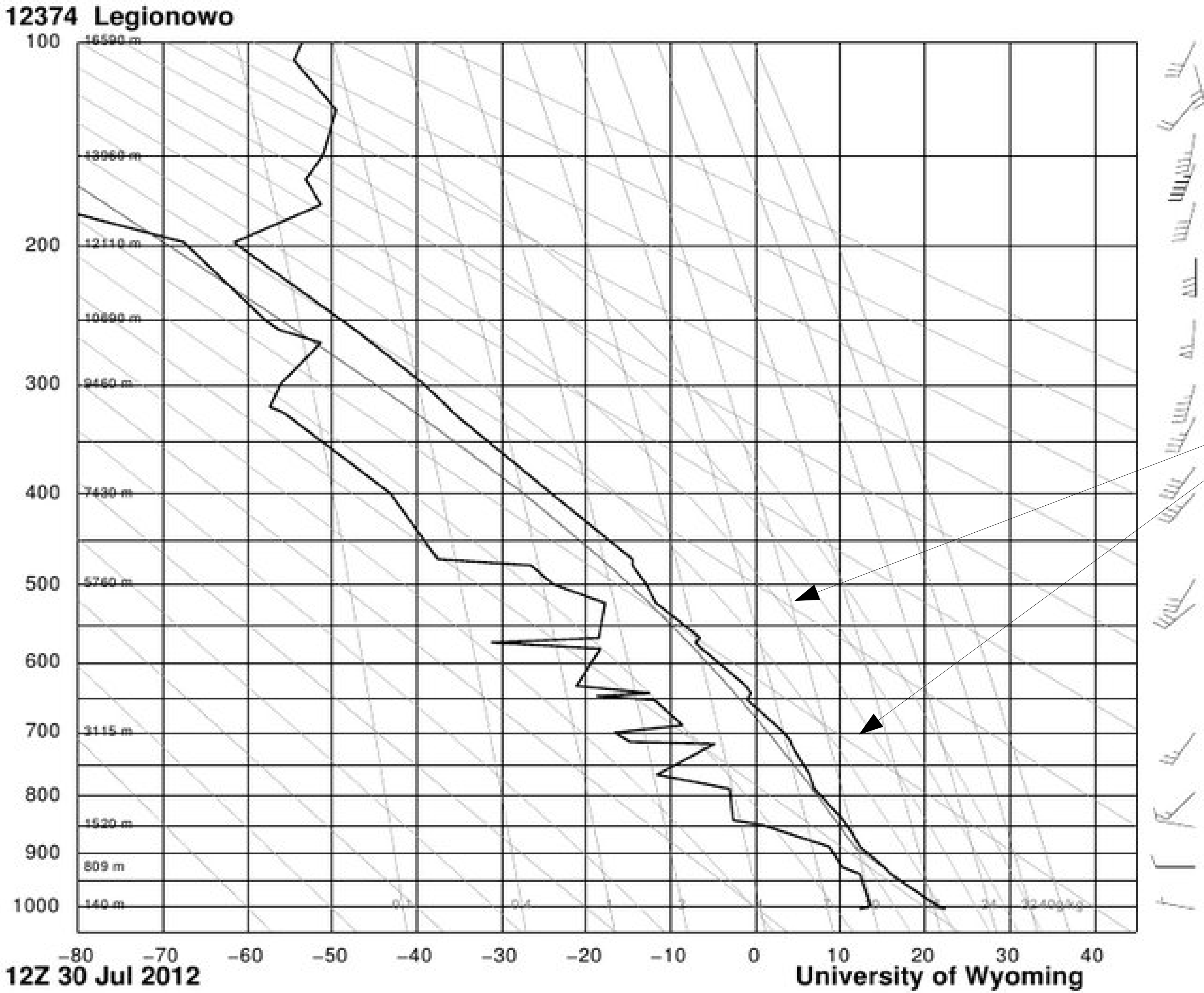
so that the dry adiabatic lapse rate is given by

$$\Gamma_d = \frac{g}{c_p}, \tag{2.229}$$

$$\text{Stability}: \qquad \frac{\partial \tilde{\theta}}{\partial z} > 0; \qquad -\frac{\partial \tilde{T}}{\partial z} < \Gamma_d \equiv \frac{g}{c_p},$$

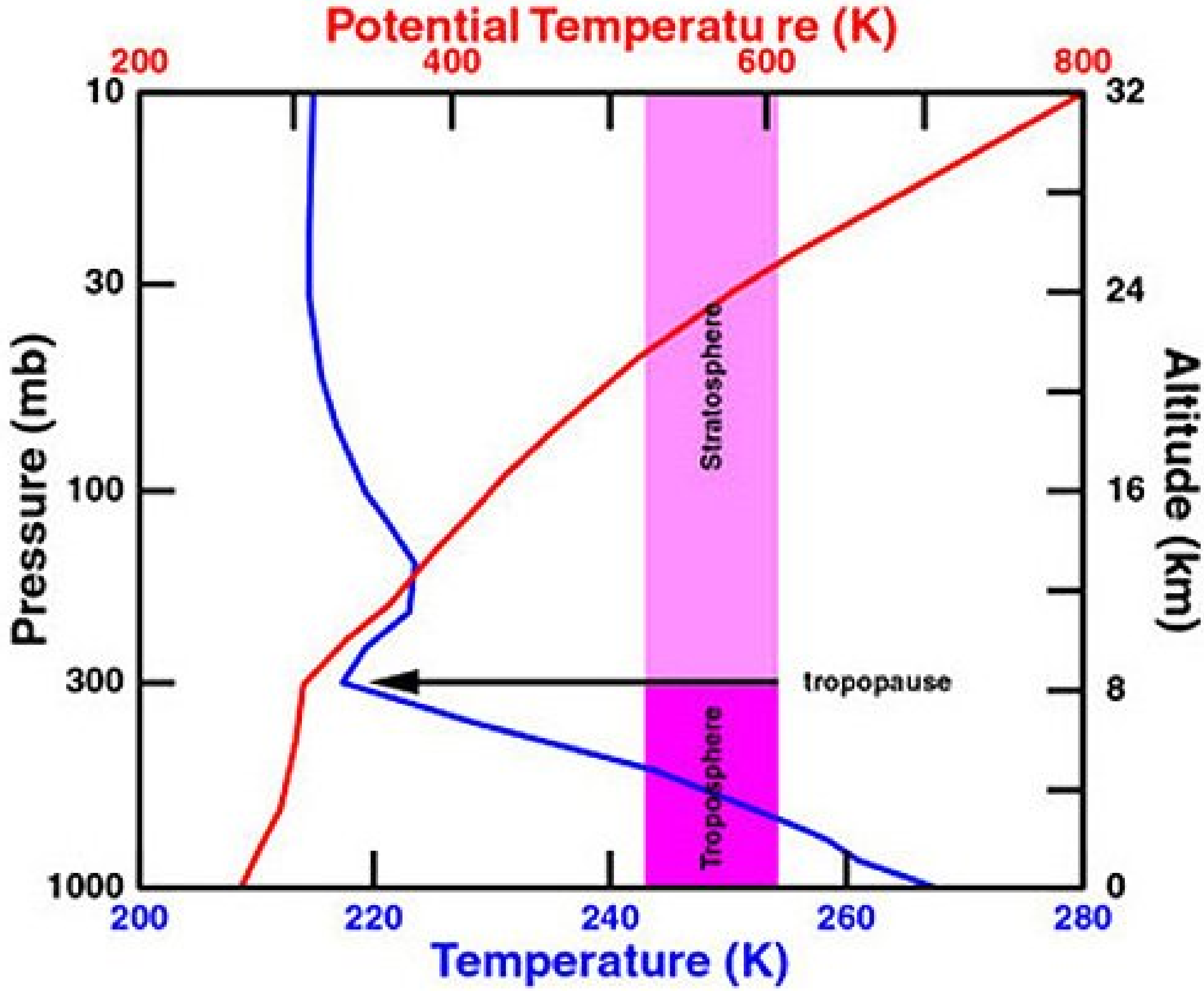
$$\text{Instability}: \qquad \frac{\partial \tilde{\theta}}{\partial z} < 0; \qquad -\frac{\partial \tilde{T}}{\partial z} > \Gamma_d \equiv \frac{g}{c_p}.$$





Adiabates

allow to deduce about stability



February 24, 1999 75°W, 40°N