Third-order accurate MPDATA for arbitrary flows

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Setting the stage

Highlights of MPDATA

- MPDATA is a sign-preserving second-order accurate scheme for numerical integration of generalised transport equation

$$\frac{\partial G\Psi}{\partial t} + \nabla \cdot (\boldsymbol{V}\Psi) = GR$$

  based on iterative application of the first-order-accurate upwind algorithm
- The basic version with two iterations uses physical velocity $\boldsymbol{V}$ in the first iteration and leading error compensating ("antidiffusive") velocity $\overline{\boldsymbol{V}}$ in the second
- Antidiffusive velocity is derived via truncation error analysis of the upwind scheme

Antidiffusive velocity on structured grids

$$\overline{\boldsymbol{V}} = \frac{1}{2}\boldsymbol{\delta x} \odot \uparrow\boldsymbol{V}\uparrow \odot \frac{\nabla\Psi}{\Psi} - \frac{1}{2}\delta t \frac{\boldsymbol{V}}{G}\left[\frac{\nabla\cdot(\boldsymbol{V}\Psi)}{\Psi} + \frac{\partial G}{\partial t}\right]$$

where

$$(\uparrow\boldsymbol{a}\uparrow)^I = |a^I|$$

denotes component-wise absolute value of a vector and

$$(\boldsymbol{a}\odot\boldsymbol{b})^I := a^I b^I$$

is the Hadamard product of two vectors

Third-order accurate MPDATA for time and space dependant flows

New idea

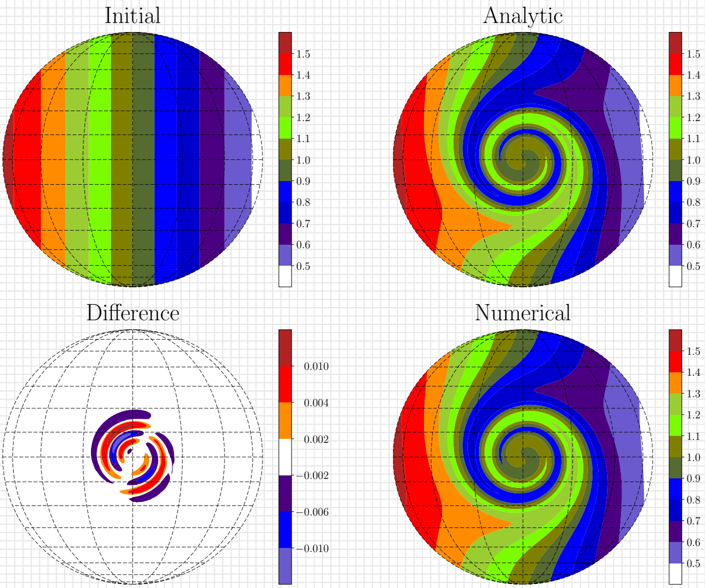
- Find "third-order" antidiffusive velocity $\overline{\overline{\boldsymbol{V}}}$ compensating MPDATA error for arbitrary flows
- Done for the constant velocity in L. Margolin, P. K. Smolarkiewicz, SISC, 1998 starting from the upwind scheme
- We take a novel approach using MPDATA as a starting point
- Consider homogeneous generalised transport equation with $G = G(\boldsymbol{x})$ and $\boldsymbol{V} = \boldsymbol{V}(t, \boldsymbol{x})$

$$G\frac{\partial \Psi}{\partial t} + \nabla \cdot (\boldsymbol{V}\Psi) = 0$$

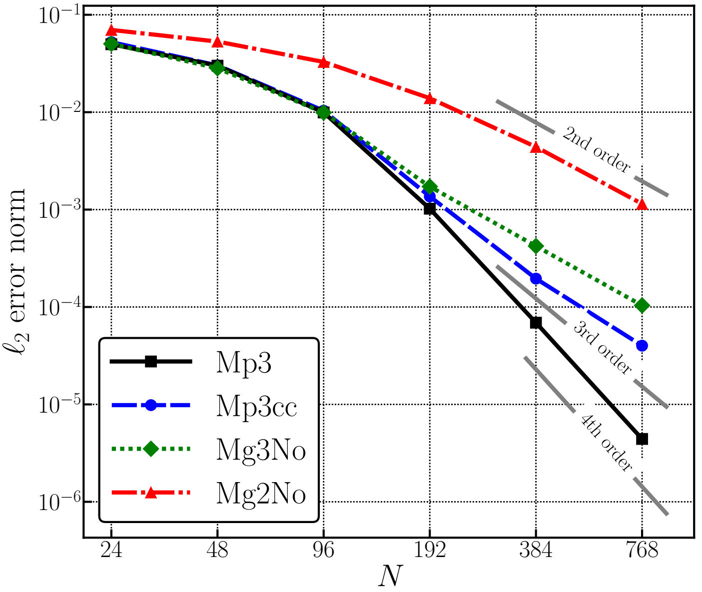
The result

$$\begin{aligned}
\overline{\overline{\boldsymbol{V}}} = &- \frac{\boldsymbol{\delta x} \odot \boldsymbol{\delta x}}{24} \odot \left[4\boldsymbol{V} \odot \frac{1}{\Psi}\nabla \odot \nabla\Psi + 2\frac{\nabla\Psi}{\Psi} \odot \nabla \odot \boldsymbol{V} + \nabla \odot \nabla \odot \boldsymbol{V} \right] \\
&+ \frac{\boldsymbol{\delta x}}{2} \odot \left\uparrow \overline{\boldsymbol{V}} \right\uparrow \odot \frac{\nabla\Psi}{\Psi} + \frac{\delta t}{2}\boldsymbol{\delta x} \odot \uparrow\boldsymbol{V}\uparrow \odot \frac{1}{\Psi}\nabla\left[\frac{1}{G}\nabla \cdot (\boldsymbol{V}\Psi)\right] \\
&+ \frac{\delta t^2}{24}\left\{ -\frac{8\boldsymbol{V}}{G\Psi}\nabla \cdot \left[\frac{\boldsymbol{V}}{G}\nabla \cdot (\boldsymbol{V}\Psi)\right] + \frac{\partial^2 \boldsymbol{V}}{\partial t^2} + \frac{2\boldsymbol{V}}{G\Psi}\nabla \cdot \left(\frac{\partial \boldsymbol{V}}{\partial t}\Psi\right) - \frac{2}{G\Psi}\frac{\partial \boldsymbol{V}}{\partial t}\nabla \cdot (\boldsymbol{V}\Psi)\right\}
\end{aligned}$$

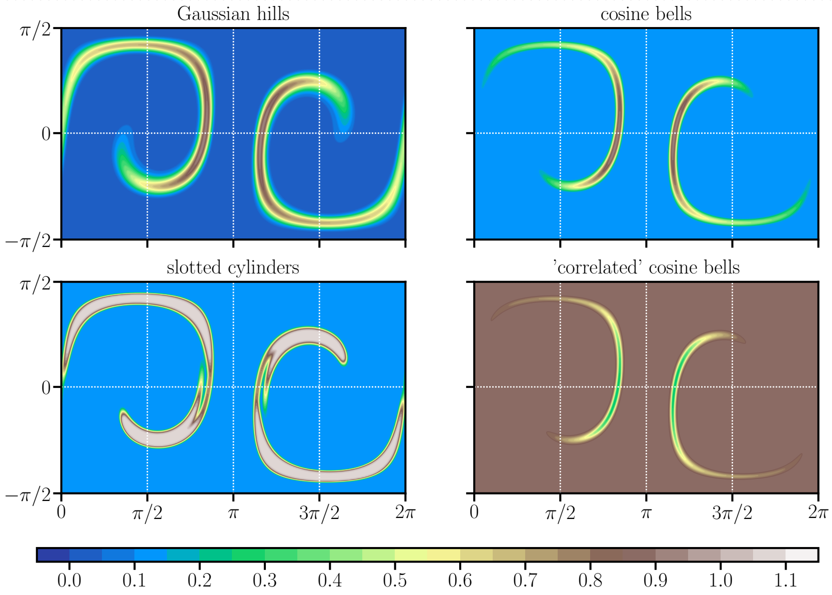
Moving vortices on the sphere – Nair & Jablonowski MWR 2008



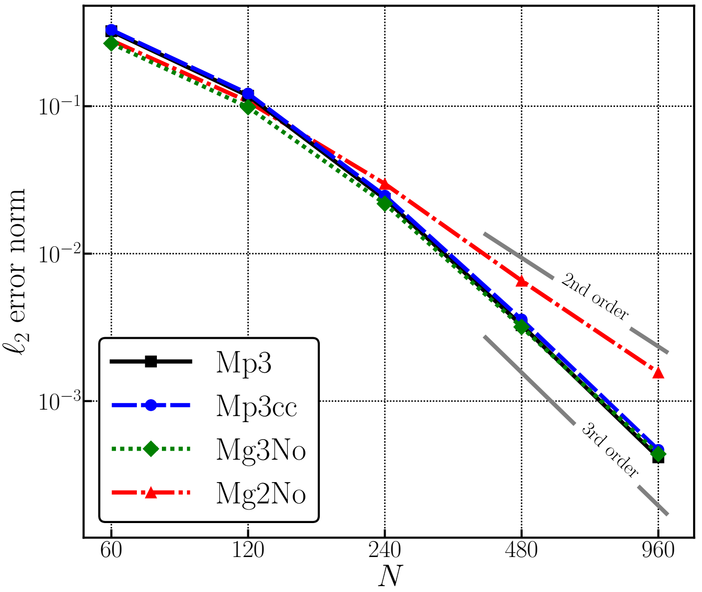
Moving vortices on the sphere – convergence results



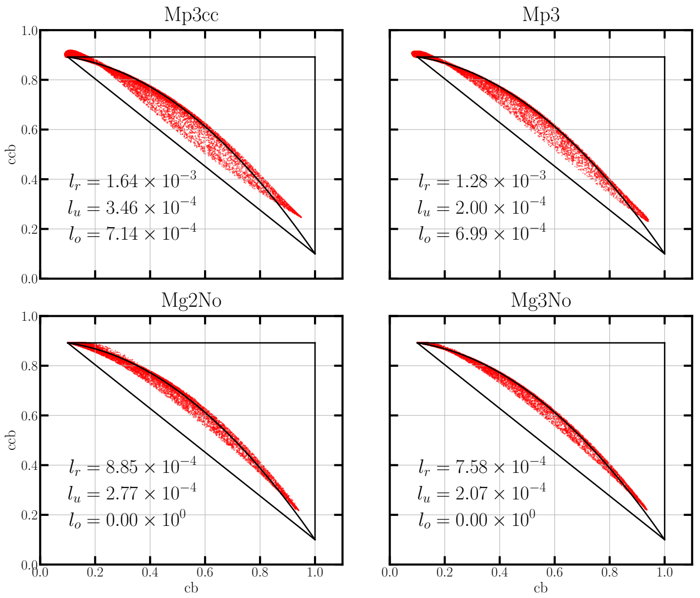
Reversing deformational flow – Lauritzen et al. GMD 2012



Reversing deformational flow – convergence results



Reversing deformational flow – numerical mixing results



Third-order accurate MPDATA coming soon to *libmpdata++*

libmpdata++

free & open source C++ library of MPDATA solvers

Repository

`https://github.com/igfuw/libmpdataxx`

Article

Jaruga et al. *libmpdata++ 0.1: a library of parallel MPDATA solvers for systems of generalised transport equations*
Geoscientific Model Development, 8, 1005-1032, 2015

Acknowledgments

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Thank you for your attention !