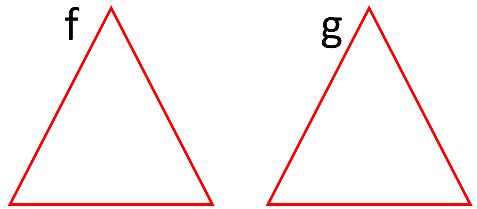
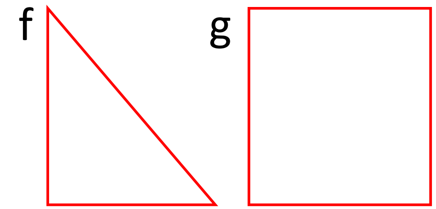
WSTĘP DO OPTYKI FOURIEROWSKIEJ

dr hab. Rafał Kasztelanic

WYKŁAD 2

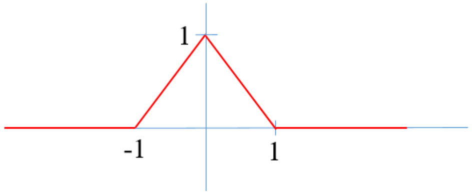
Splot





Splot

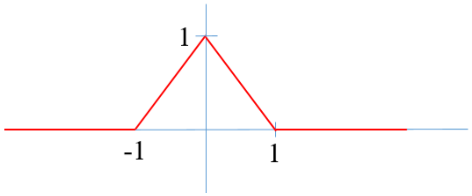
$$\Lambda(x) = \begin{cases} 1 - |x| & |x| < 1 \\ 0 & |x| \geq 1 \end{cases}$$



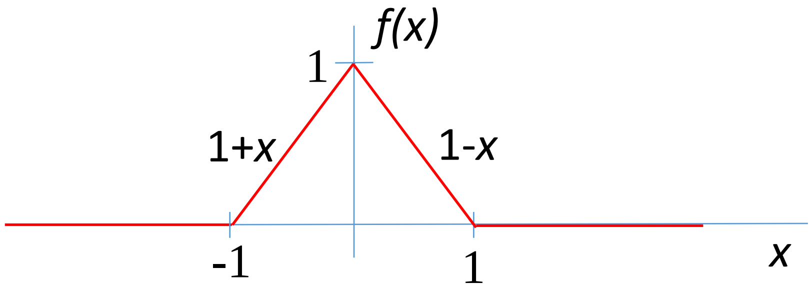
$$h(x) = \int_{-\infty}^{\infty} f(x - x')\, g(x')\, \mathrm{d}x'$$

Splot

$$\Lambda(x) = \begin{cases} 1 - |x| & |x| < 1 \\ 0 & |x| \geq 1 \end{cases}$$

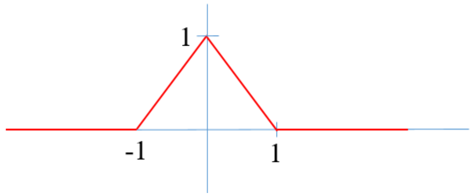


$$h(x) = \int_{-\infty}^{\infty} f(x - x')\, g(x')\, dx'$$

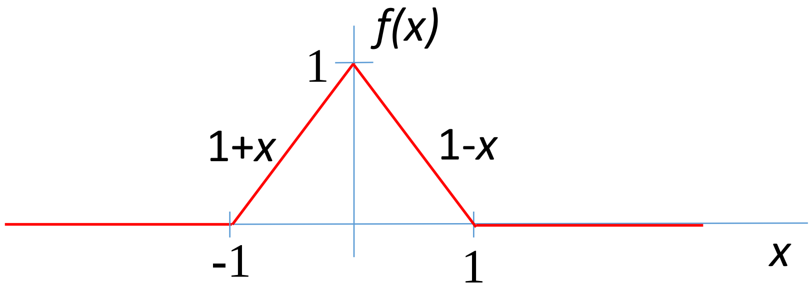


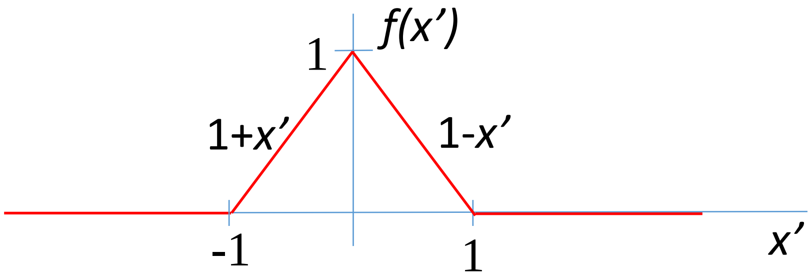
Splot

$$\Lambda(x) = \begin{cases} 1 - |x| & |x| < 1 \\ 0 & |x| \geq 1 \end{cases}$$



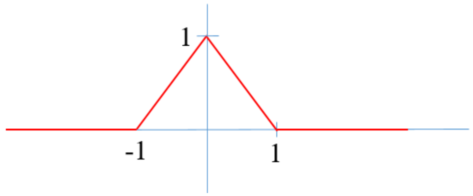
$$h(x) = \int_{-\infty}^{\infty} f(x - x')\, g(x')\, \mathrm{d}x'$$



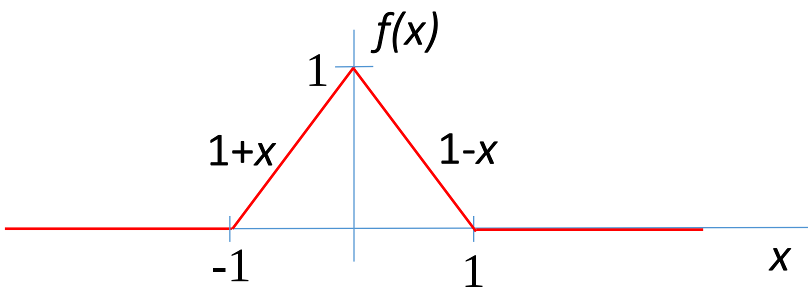


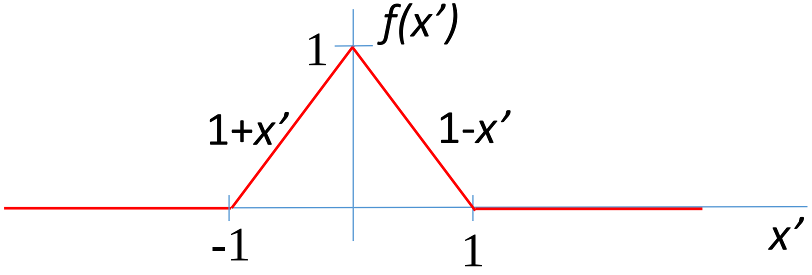
Splot

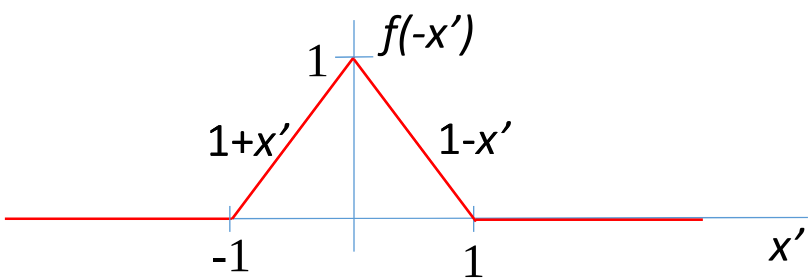
$$\Lambda(x) = \begin{cases} 1 - |x| & |x| < 1 \\ 0 & |x| \geq 1 \end{cases}$$



$$h(x) = \int_{-\infty}^{\infty} f(x - x') g(x') \, dx'$$

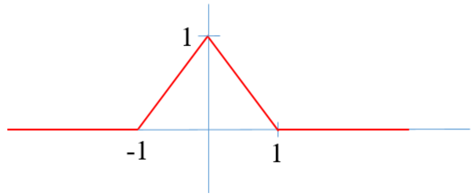




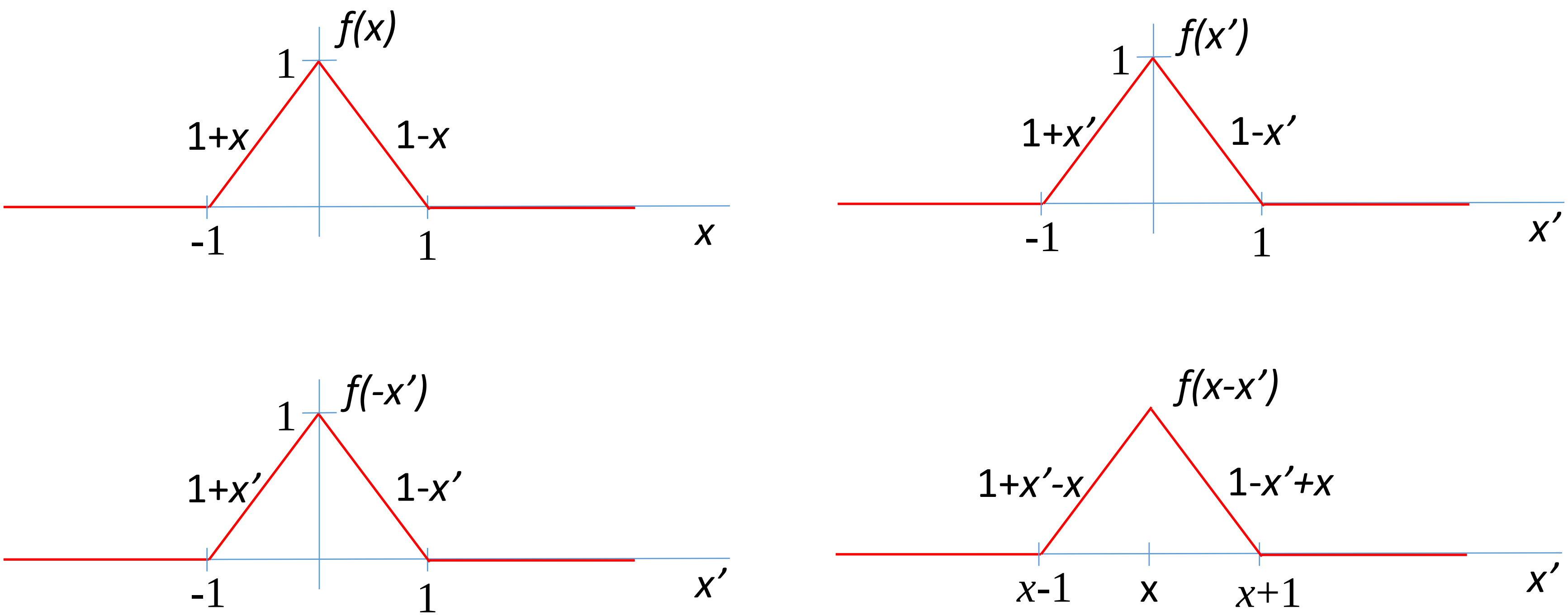


Splot

$$\Lambda(x) = \begin{cases} 1 - |x| & |x| < 1 \\ 0 & |x| \geq 1 \end{cases}$$

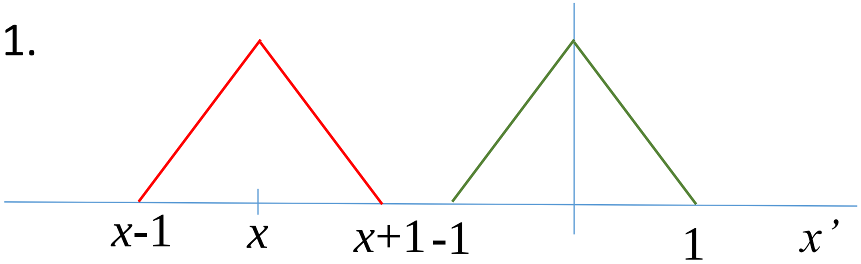


$$h(x) = \int_{-\infty}^{\infty} f(x - x')\, g(x')\, dx'$$



Splot

1.

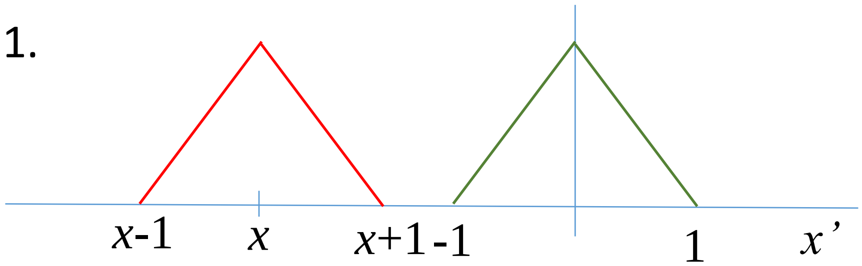


$$x + 1 < -1 \quad \rightarrow \quad x < -2$$

$$\boxed{h(x) = 0}$$

Splot

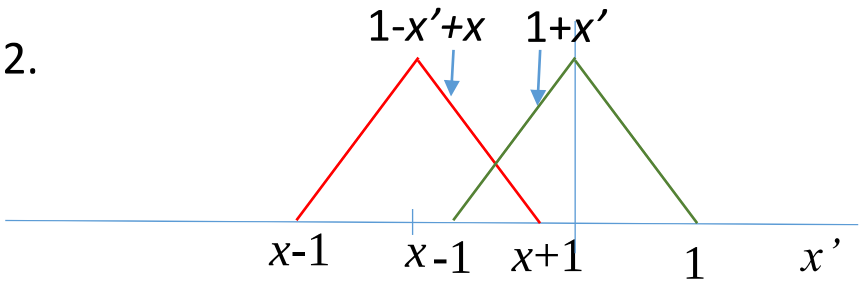
1.



$$x + 1 < -1 \;\; \rightarrow \;\; x < -2$$

$$\boxed{h(x) = 0}$$

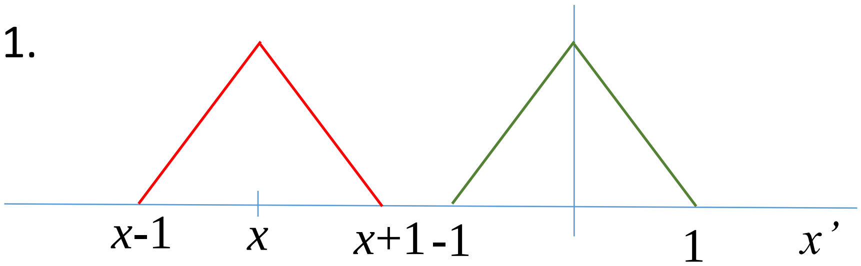
2.



$$-1 < x + 1 < 0 \;\; \rightarrow \;\; -2 < x < -1$$

Splot

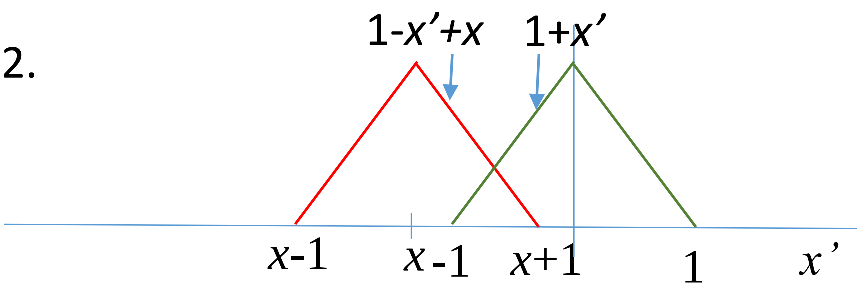
1.



$$x+1<-1 \;\rightarrow\; x<-2$$

$$\boxed{h(x)=0}$$

2.



$$-1<x+1<0 \;\rightarrow\; -2<x<-1$$

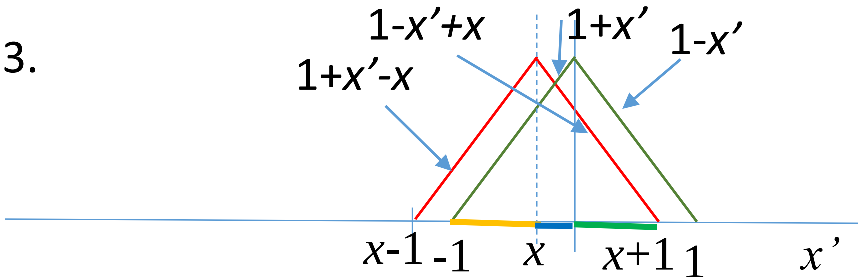
$$h(x)=\int_{-1}^{x+1}(1+x')(1-x'+x)\,dx'=$$

$$=\int_{-1}^{x+1}\left(1+x-x'^2+xx'\right)dx'=(1+x)\underbrace{\int_{-1}^{x+1}dx'}_{-(x+1)(x+2)}+x\underbrace{\int_{-1}^{x+1}x'dx'}_{-\frac{x}{2}\left((x+1)^2-1\right)}-\underbrace{\int_{-1}^{x+1}x'^2dx'}_{-\frac{1}{3}-\frac{(x+1)^3}{3}}=$$

$$\boxed{h(x)=\frac{1}{6}(x+2)^3}$$

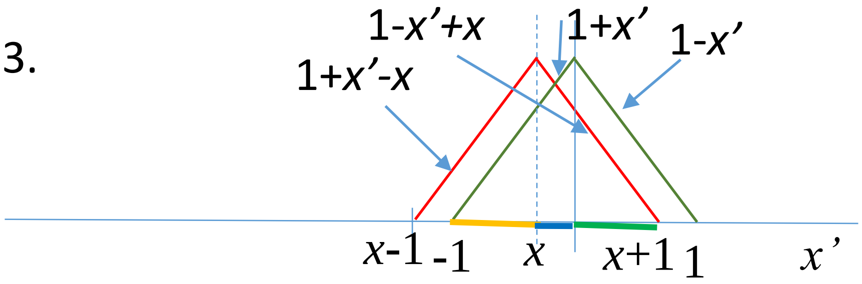
Splot

3.



$$0 < x + 1 < 1 \;\; \rightarrow \;\; -1 < x < 0$$

3.



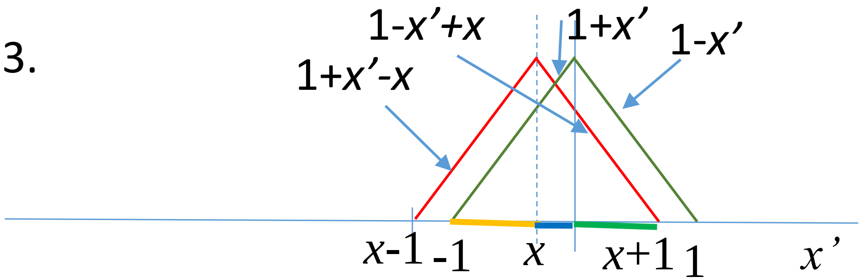
$$0 < x + 1 < 1 \;\; \rightarrow \;\; -1 < x < 0$$

$$h(x) = \underbrace{\int_{-1}^{x} (1 + x')(1 + x' - x)\, dx'}_{\frac{1}{6}(-x^3 + 3x + 2)} + \underbrace{\int_{x}^{0} (1 + x')(1 - x' + x)\, dx'}_{-\frac{1}{6}x(x^2 + 6x + 6)} + \underbrace{\int_{0}^{x+1} (1 - x')(1 - x' + x)\, dx'}_{\frac{1}{6}x(-x^3 + 3x + 2)}$$

$$\boxed{h(x) = \frac{1}{3}(-x^3 + 3x + 2) - \frac{1}{6}x(x^2 + 6x + 6)}$$

Splot

3.



$$0 < x+1 < 1 \quad \rightarrow \quad -1 < x < 0$$

$$h(x) = \underbrace{\int_{-1}^{x}(1+x')(1+x'-x)\,dx'}_{\frac{1}{6}(-x^3+3x+2)} + \underbrace{\int_{x}^{0}(1+x')(1-x'+x)\,dx'}_{-\frac{1}{6}x(x^2+6x+6)} + \underbrace{\int_{0}^{x+1}(1-x')(1-x'+x)\,dx'}_{\frac{1}{6}x(-x^3+3x+2)}$$

$$h(x) = \frac{1}{3}(-x^3+3x+2) - \frac{1}{6}x(x^2+6x+6)$$

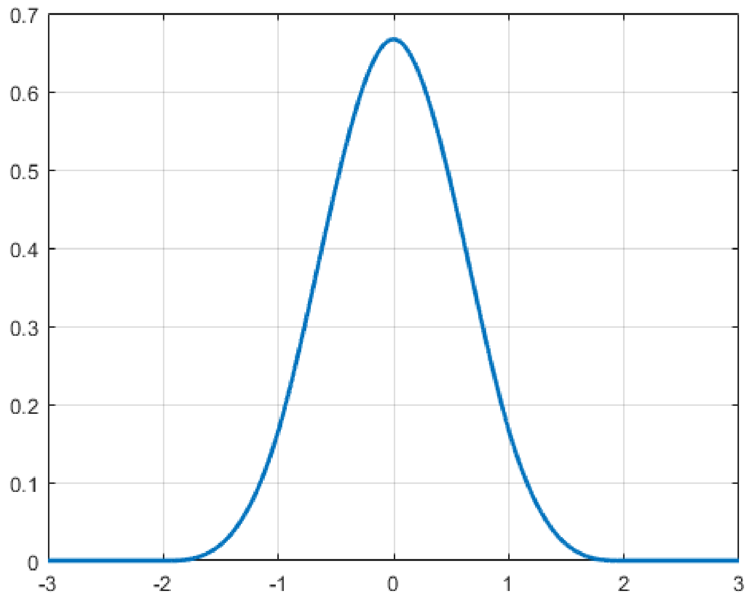
4. $-1 < x-1 < 0 \quad \rightarrow \quad 0 < x < 1$ Symetrycznie do 3

5. $0 < x-1 < 1 \quad \rightarrow \quad 1 < x < 2$ Symetrycznie do 2

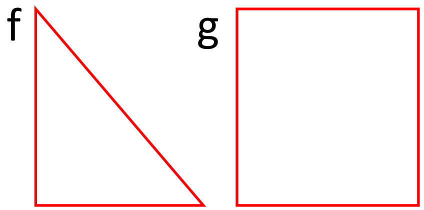
6. $1 < x-1 \quad \rightarrow \quad 2 < x$ Symetrycznie do 1

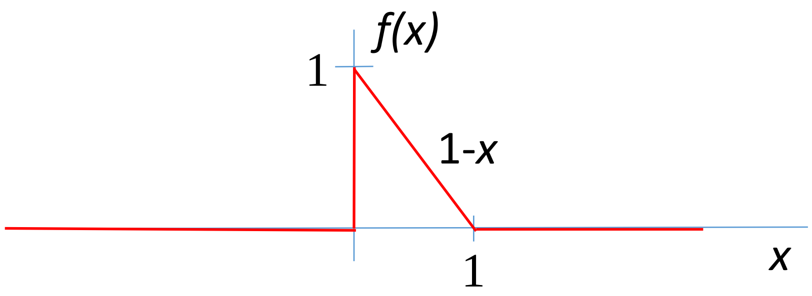
Splot

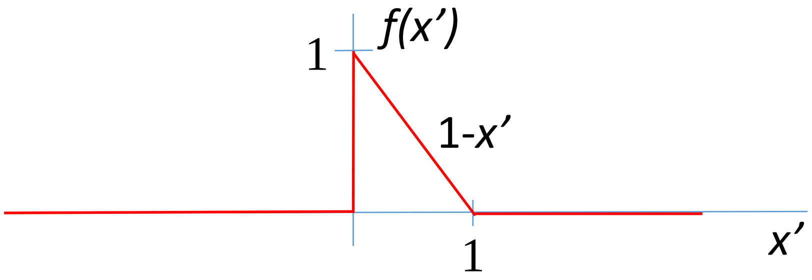
$$h(x) = \begin{cases} 0 & x < -2 \\ \frac{1}{6}(x+2)^3 & -2 < x < -1 \\ \frac{1}{3}(-x^3+3x+2) - \frac{1}{6}x(x^2+6x+6) & -1 < x < 0 \\ \frac{1}{3}(x^3-3x+2) + \frac{1}{6}x(x^2-6x+6) & 0 < x < 1 \\ -\frac{1}{6}(x-2)^3 & 1 < x < 2 \\ 0 & 2 < x \end{cases}$$

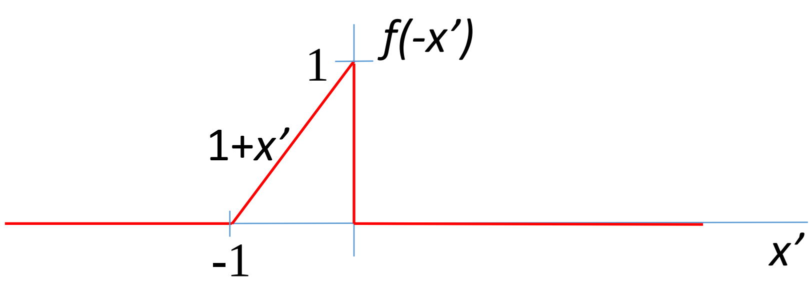


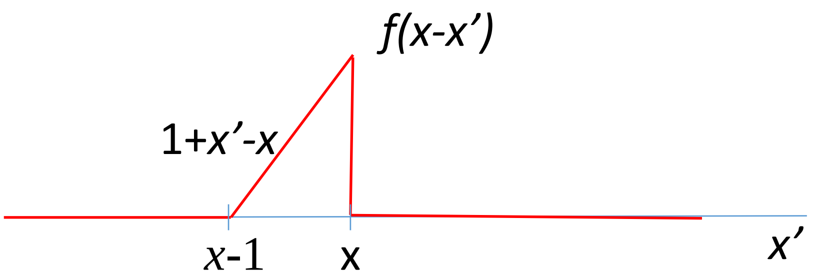
Splot





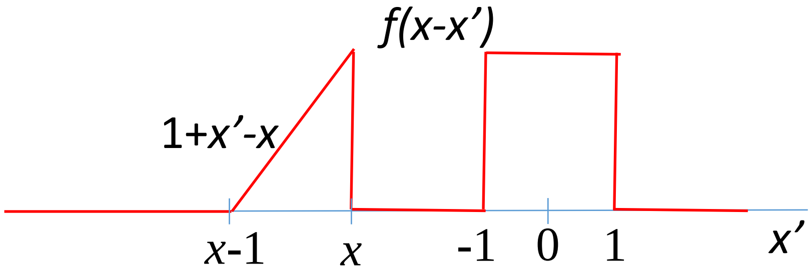






Splot

1.

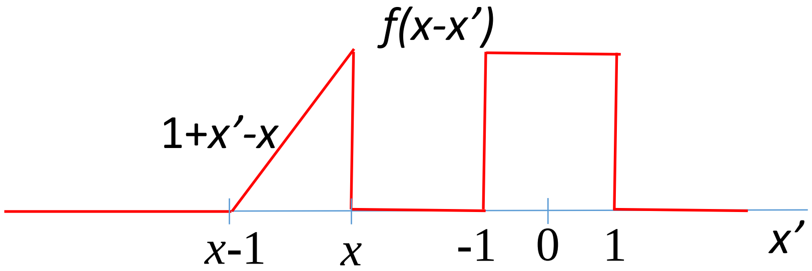


$x < -1$

$$h(x) = 0$$

Splot

1.



$x < -1$

$$h(x) = 0$$

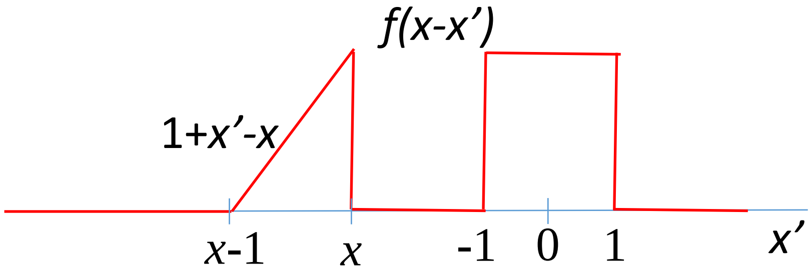
2. $-1 < x < 0$

$$h(x) = \int_{-1}^{x} (1 + x' - x)dx' = (1 - x) \int_{-1}^{x} dx' + \int_{-1}^{x} x'dx' =$$

$$= (1 - x)(x + 1) + \frac{x^2}{2} + \frac{1}{2} = \frac{1}{2}(1 - x^2)$$

Splot

1.



$x < -1$

$$h(x) = 0$$

2. $-1 < x < 0$

$$h(x) = \int_{-1}^{x} (1 + x' - x)dx' = (1 - x)\int_{-1}^{x} dx' + \int_{-1}^{x} x'dx' =$$

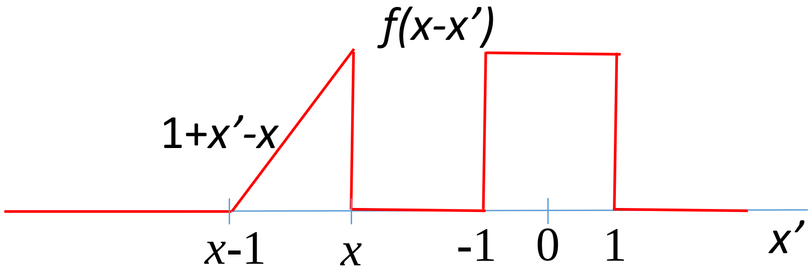
$$= (1 - x)(x + 1) + \frac{x^2}{2} + \frac{1}{2} = \frac{1}{2}(1 - x^2)$$

3. $0 < x < 1$

$$h(x) = \frac{1}{2}$$

Splot

1.



$x < -1$

$$h(x) = 0$$

2. $-1 < x < 0$

$$h(x) = \int_{-1}^{x} (1 + x' - x)dx' = (1 - x)\int_{-1}^{x} dx' + \int_{-1}^{x} x'dx' =$$

$$= (1 - x)(x + 1) + \frac{x^2}{2} + \frac{1}{2} = \frac{1}{2}(1 - x^2)$$

3. $0 < x < 1$

$$h(x) = \frac{1}{2}$$

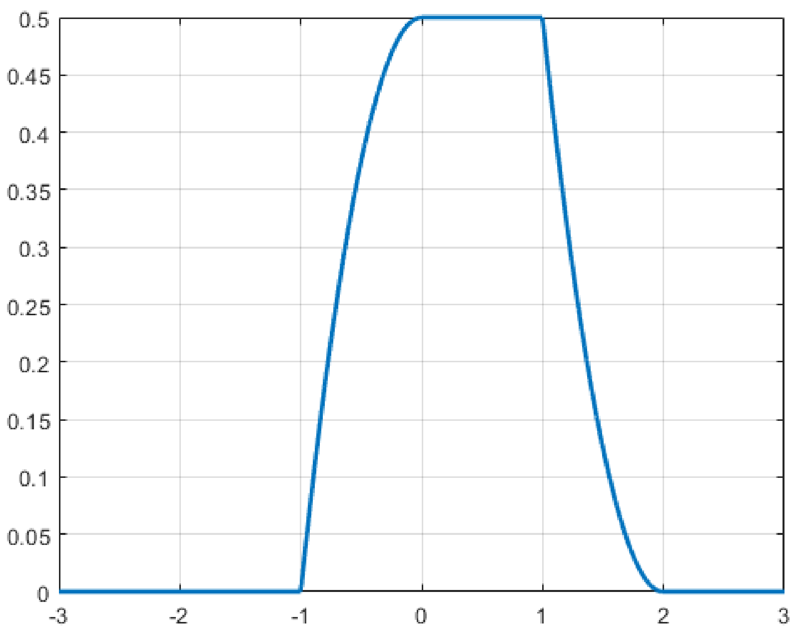
4. $0 < x - 1 < 1 \quad \rightarrow \quad 1 < x < 2$

$$h(x) = \int_{x-1}^{1} (1 + x' - x)dx' = (1 - x)\int_{x-1}^{1} dx' + \int_{x-1}^{1} x'dx' =$$

$$= (1 - x)(2 - x) + \frac{1}{2}[1 - (x - 1)^2] = \frac{x^2}{2} - 2x + 2$$

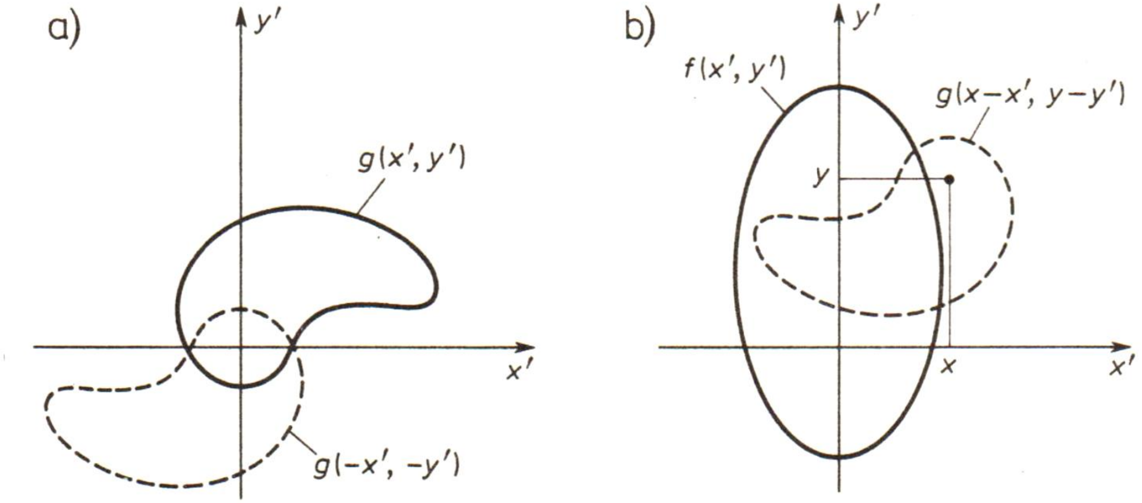
Splot

$$h(x) = \begin{cases} 0 & x < -1 \\ \frac{1}{2}(1 - x^2) & -1 < x < 0 \\ \frac{1}{2} & 0 < x < 1 \\ \frac{x^2}{2} - 2x + 2 & 1 < x < 2 \\ 0 & 2 < x \end{cases}$$



Splot.m

Splot 2D



Korelacja

$$\varphi(x) = \int_{-\infty}^{\infty} f(x')g(x' - x)dx'$$

$$\varphi(x) = f(x) \star g(x)$$

Splot:

$$h(x) = \int_{-\infty}^{\infty} f(x - x')g(x')\,\mathrm{d}x'$$

Korelacja

$$\varphi(x) = \int_{-\infty}^{\infty} f(x')g(x' - x)dx'$$

$$\varphi(x) = f(x) \star g(x)$$

Splot:

$$h(x) = \int_{-\infty}^{\infty} f(x - x')g(x')\,dx'$$

Wasności:

Korelacja nie jest przemienna:

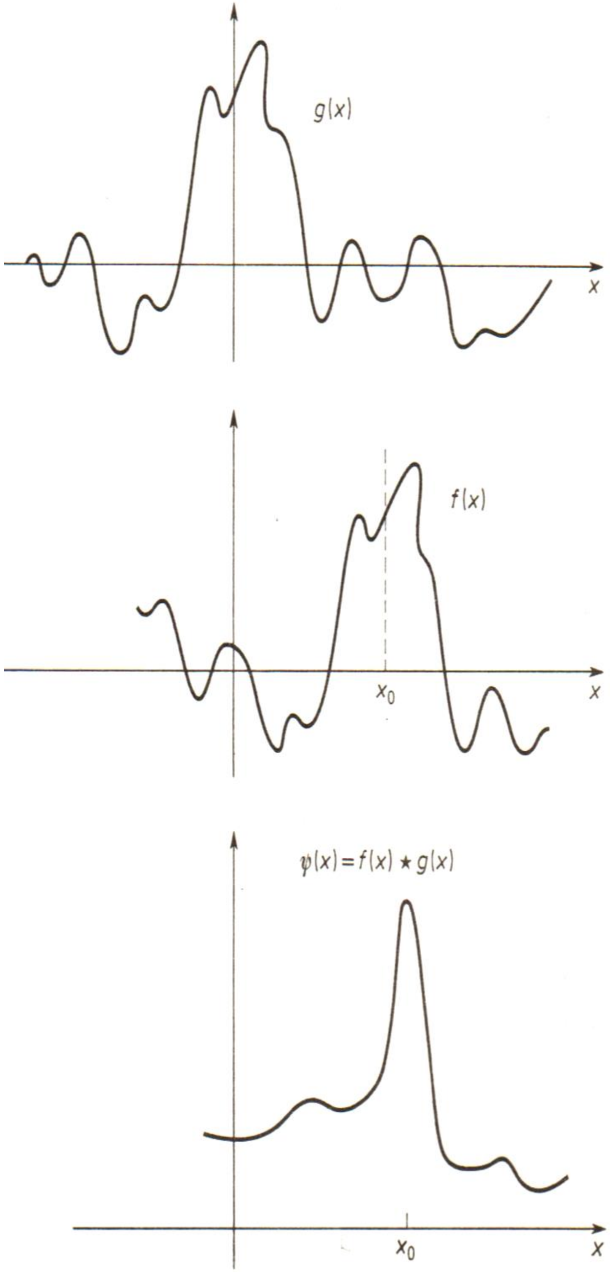
$$f(x) \star g(x) \neq g(x) \star f(x)$$

Korelacja jest równa splotowi z odwróconą funkcją *g*:

$$f(x) \star g(x) = f(x) \otimes g(-x)$$

Gdy g(x) jest funkcją parzystą to korelacja jest równoważna splotowi

Korelacja



Autokorelacja

Autokorelacja gdy $g(x) = f(x)$

Współczynnik autokorelacji: $\gamma(x) = \dfrac{\varphi(x)}{\varphi(0)}$

Moduł autokorelacji osiąga największą wartość w ‚0' : $|\varphi(x)| \leq \varphi(0)$

Szereg Taylora:

$$\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ \dots = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^{2n-1}}{(2n-1)!}$$

SzeregTaylora.mlx

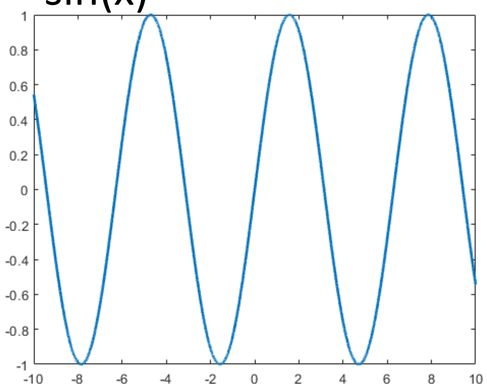
Transformacja Fouriera

Szereg Taylora:

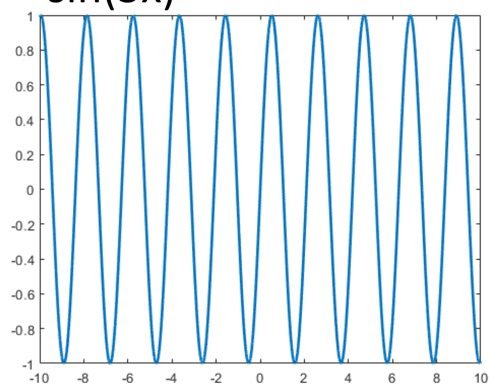
$$\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ \dots = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^{2n-1}}{(2n-1)!}$$

Suma sinusów:

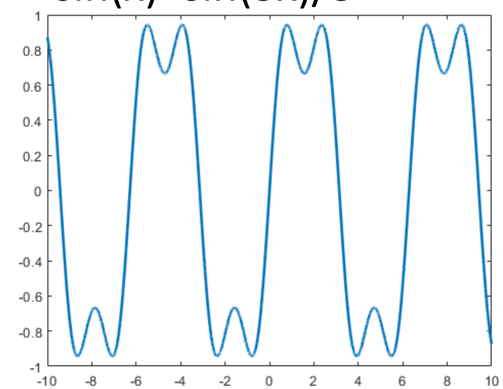
sin(x)



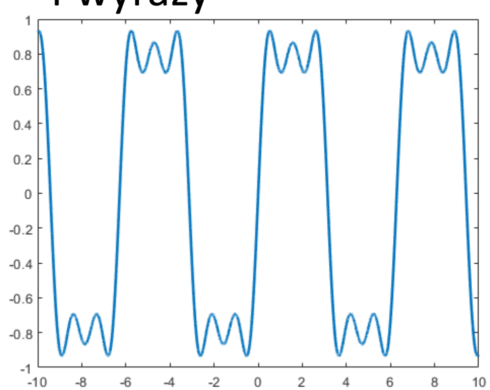
sin(3x)



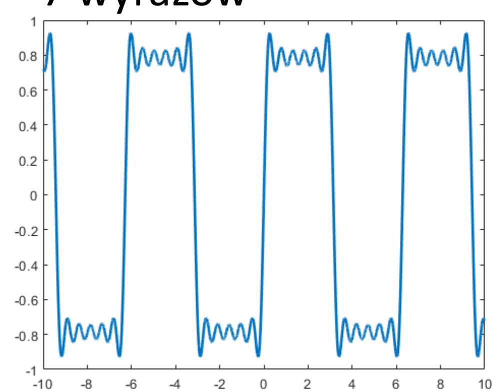
sin(x)+sin(3x)/3



4 wyrazy



7 wyrazów



$$\sum_{n=1}^{N} \frac{\sin[(2n-1)x]}{2n-1}$$

ProstokatSzereg.mlx

Transformacja Fouriera

Funkcję periodyczną można przedstawić jako sumę sinusów:

$$f(t)=\sum_{k=1}^{n}[A_k \sin(2\pi\omega_k t + \pi/2) + B_k \sin(2\pi\omega_k t)]$$

Wygodniej to przepisać jako sumę sinusa i cosinusa:

$$f(t)=\sum_{k=1}^{n}[A_k \cos(2\pi\omega_k t) + B_k \sin(2\pi\omega_k t)]$$

Np. dla funkcji:

$$f_1 = \frac{1}{2}\sin(\pi t) + 2\sin(4\pi t) + 4\cos(2\pi t)$$

amplitudy

częstości

Transformacja Fouriera

Zapis zespolony funkcji trygonometrycznych:

$$e^{i\varphi} = \cos(\varphi) + i\sin(\varphi) \qquad e^{-i\varphi} = \cos(\varphi) - i\sin(\varphi)$$

$$\cos(\varphi) = \frac{e^{i\varphi} + e^{-i\varphi}}{2} \qquad \sin(\varphi) = \frac{e^{i\varphi} - e^{-i\varphi}}{2}$$

Czyli $f(t) = \sum_{k=1}^{n}[A_k \cos(2\pi\omega_k t) + B_k \sin(2\pi\omega_k t)]$ mogę zapisać jako:

$$f(t) = \sum_{k=1}^{n}\left[\frac{A_k}{2}\left(e^{2\pi i\omega_k t} + e^{-2\pi i\omega_k t}\right) + \frac{B_k}{2}\left(e^{2\pi i\omega_k t} - e^{-2\pi i\omega_k t}\right)\right]$$

Podstawiam:

$$C_k = \begin{cases} \frac{A_k - iB_k}{2} & dla \quad k > 0 \\ \frac{A_k + iB_k}{2} & dla \quad k < 0 \end{cases} \qquad \omega_k = \omega_{-k} \quad dla \quad k < 0$$

Transformacja Fouriera

Dostaję:

$$f(t)=\sum_{k=-n}^{n}[C_k e^{2\pi i \omega_k t}]$$

Czyli nasza funkcja:

$$f_1 = \frac{1}{2}\sin(\pi t) + 2\sin(4\pi t) + 4\cos(2\pi t)$$

k	Częstotliwość (ω_k)	C_k
3	2	1
2	1	2
1	1/2	1/4
0	0	0
-1	-1/2	-1/4
-2	-1	2
-3	-2	-1

Transformacja Fouriera

$$f_1 = \boxed{\frac{1}{2}\sin(\pi t)} + 2\sin(4\pi t) + 4\cos(2\pi t)$$

$$C_{-k}\exp[2\pi i\omega t] + C_k\exp[-2\pi i\omega t]$$

$$C_{-1}\exp[2\pi i\omega t] + C_1\exp[-2\pi i\omega t] =$$

$$C\exp[2\pi i\omega t] - C\exp[-2\pi i\omega t] =$$

$$\sin(\varphi) = \frac{e^{i\varphi} - e^{-i\varphi}}{2}$$

$$\boxed{2C}\boxed{\left[\frac{\exp[2\pi i\omega t] - \exp[-2\pi i\omega t]}{2}\right]}$$

$2C \rightarrow \frac{1}{2} \rightarrow C = \frac{1}{4}$

$\left[\frac{\exp[2\pi i\omega t] - \exp[-2\pi i\omega t]}{2}\right] \rightarrow \sin(\pi t) \rightarrow \omega = \frac{1}{2}$

k	Częstotliwość (ω_k)	C_k
3	2	1
2	1	2
1	1/2	1/4
0	0	0
-1	-1/2	-1/4
-2	-1	2
-3	-2	-1

Transformacja Fouriera

$$f_1 = \frac{1}{2}\sin(\pi t) + 2\sin(4\pi t) + 4\cos(2\pi t)$$

$$C_{-k}\exp[2\pi i\omega t] + C_k\exp[-2\pi i\omega t]$$

$$C_{-2}\exp[2\pi i\omega t] + C_2\exp[-2\pi i\omega t] =$$

$$C\exp[2\pi i\omega t] + C\exp[-2\pi i\omega t] =$$

$$\cos(\varphi) = \frac{e^{i\varphi} + e^{-i\varphi}}{2}$$

$$2C\left[\frac{\exp[2\pi i\omega t] - \exp[-2\pi i\omega t]}{2}\right]$$

4 → $C = 2$

$\cos(2\pi t)$ → $\omega = 1$

k	Częstotliwość (ω_k)	C_k
3	2	1
2	1	2
1	1/2	1/4
0	0	0
-1	-1/2	-1/4
-2	-1	2
-3	-2	-1

Transformacja Fouriera

$$f_1 = \frac{1}{2}\sin(\pi t) + 2\sin(4\pi t) + 4\cos(2\pi t)$$

$$C_{-k}\exp[2\pi i\omega t] + C_k\exp[-2\pi i\omega t]$$

$$C_{-3}\exp[2\pi i\omega t] + C_3\exp[-2\pi i\omega t] =$$

$$C\exp[2\pi i\omega t] - C\exp[-2\pi i\omega t] =$$

$$\sin(\varphi) = \frac{e^{i\varphi} - e^{-i\varphi}}{2}$$

$$2C\left[\frac{\exp[2\pi i\omega t] - \exp[-2\pi i\omega t]}{2}\right]$$

$2 \rightarrow C = 1$

$\cos(4\pi t) \rightarrow \omega = 2$

k	Częstotliwość (ω_k)	C_k
3	2	1
2	1	2
1	1/2	1/4
0	0	0
-1	-1/2	-1/4
-2	-1	2
-3	-2	-1

Transformacja Fouriera

Przechodzimy do funkcji ciągłej:

$f(\text{t})=\int_{-\infty}^{\infty} F(\omega)\, e^{2\pi i\omega t}\text{d}\omega$ **ODWROTNA TRANSFORMATA FOURIERA**

Składowe częstości ~ C_k

$F(\text{t})=\int_{-\infty}^{\infty} f(\omega)\, e^{-2\pi i\omega t}\text{d}\omega$ **TRANSFORMATA FOURIERA**