Hydrodynamics and elasticity (Gustavo Abade and Daniel Albuquerque)

Homework 2

Due October 29, 2020

Solutions, coments and corrections should be sent to d.albuquerque@uw.edu.pl

1. Consider the following steady 2D flow: $\mathbf{u} = (ky, -kx)$,

   where k is a real constant.

   (i) Check, whether this flow is solenoidal (e.g. divergence-free). Find the equations for the streamlines and sketch them. Find the vorticity.

   (ii) Imagine that at $t = 0$ particles positioned on a circle of radius a around the origin, given by $x^2 + y^2 = a^2$, were marked with a colored dye. Find the equation for this material curve at any time $t > 0$. How does the surface area enclosed by this curve change in time?

2. Lagrangean approach. Consider the following two-dimensional flow defined in Cartesian system:

$$\mathbf{u} = (kx, -ky) \tag{1}$$

   where k is a constant. Let the local concentration $c(\mathbf{r}, t)$ of a certain pollutant in this flow be described by

$$c(x, y, t) = \beta x^2 y \exp(-kt),$$

   where $y > 0$ and β is a constant. Find the rate of change in time of the substance concentration in the element of fluid moving in the velocity field (1).

3. The path of a fluid particle, that was at $\mathbf{R} = (X, Y, Z)$ at time $t = 0$, is described by $\mathbf{r}(t) = [x(t), y(t), z(t)]$ with

$$x(t) = \left\{ X \cos\left[\frac{\Omega}{2\alpha}(e^{2\alpha t} - 1)\right] - Y \sin\left[\frac{\Omega}{2\alpha}(e^{2\alpha t} - 1)\right] \right\} e^{-\alpha t},$$

$$y(t) = \left\{ Y \cos\left[\frac{\Omega}{2\alpha}(e^{2\alpha t} - 1)\right] + X \sin\left[\frac{\Omega}{2\alpha}(e^{2\alpha t} - 1)\right] \right\} e^{-\alpha t},$$

$$z(t) = Z e^{2\alpha t}.$$

(a) Show that the flow velocity reads

$$\mathbf{u} = \left(-\alpha x - \Omega y e^{2\alpha t}, -\alpha y + \Omega x e^{2\alpha t}, 2\alpha z\right), \tag{2}$$

where Ω and α are constants. Verify that $\nabla \cdot \mathbf{u} = 0$.

(b) Consider a material curve, described by the equation

$$X^2 + Y^2 = a^2, \qquad Z = 0,$$

at time $t = 0$. Obtain the equation that describes the shape of this material curve at any time $t > 0$.

Hint. Note that the velocity field in Eq. (2) has the form $\mathbf{u} = \mathbf{u}_1 + \mathbf{u}_2$, where

$$\mathbf{u}_1 = \left(-\alpha x, -\alpha y, 2\alpha z\right), \quad \mathbf{u}_2 = \left(-\Omega y e^{2\alpha t}, \Omega x e^{2\alpha t}, 0\right),$$

$\nabla \times \mathbf{u}_1 = \mathbf{0}$ and $\mathbf{u}_2$ is a rigid body motion about the z axis.

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