Session 12

1. *Internal buoyancy waves*. Consider the linearized equations of motion in two dimensions in the vertical plane (x, z) under the Boussinesq approximation. The statically stable base state is characterized by $\{\rho_0 = \text{const.}, \bar{p}(z), \bar{\theta}(z)\}$, with

$$\frac{\mathrm{d}\bar{p}}{\mathrm{d}z} = -\rho_0 g,$$

$$\bar{\theta} = \frac{\bar{p}}{\rho_0 R}\left(\frac{p_R}{\bar{p}}\right)^{\kappa},$$

and a constant zonal flow $\bar{\mathbf{u}} = (\bar{u}, 0)$. One can show that a small vertical velocity fluctuation w' satisfies the following equation:

$$\left(\frac{\partial}{\partial t} + \bar{u}\frac{\partial}{\partial x}\right)^2 \left(\frac{\partial^2 w'}{\partial x^2} + \frac{\partial^2 w'}{\partial z^2}\right) + N^2 \frac{\partial^2 w'}{\partial x^2} = 0, \tag{1}$$

where

$$N^2 = g\frac{\mathrm{d}\ln\bar{\theta}}{\mathrm{d}z}, \tag{2}$$

is the buoyancy frequency of the base state (assumed to be constant).

(a) Assume that w' has a form of a plane wave

$$w'(x, z, t) = \mathrm{Re}\{\tilde{W}\exp[i(k_x x + k_z z - \omega t)]\}, \tag{3}$$

where $\tilde{W} = We^{i\phi}$ is a complex-valued amplitude,and ϕ is a constant reference phase. Obtain the dispersion relation $\omega(k_x, k_z)$,

$$(\omega - \bar{u}k_x)^2 \left(k_x^2 + k_z^2\right) - N^2 k_x^2 = 0, \tag{4}$$

or in terms of the intrinsic frequency, $\hat{\omega} \equiv \omega - \bar{u}k_x$,

$$\hat{\omega} = \pm N \frac{k_x}{(k_x^2 + k_z^2)^{1/2}}. \tag{5}$$

(b) Take $k_x > 0$, $k_z < 0$ and chose the positive root in Eq. (5). Evaluate the phase speed, the group velocity, and show that the group velocity is perpendicular to the direction of the phase propagation. Make a sketch of the wave crests.

(c) Show that the intrinsic gravity wave frequency must be less than the buoyancy frequency. What would happen if we agitate a stratified fluid at a frequency greater than N?

2. *Topographic waves*. Consider the situation where a zonal wind with velocity $\bar{u}$ flows over a sinusoidal pattern of ridges under statically stable conditions. Consider only *stationary waves* $w'(x, z)$ (no t-dependence!) relative to the ground and obtain the dispersion relation. Under which conditions the wave generated by the topography propagates vertically (i.e., it is not vertically trapped)?