Dynamics of the Atmosphere and the Ocean

Lecture 2/3

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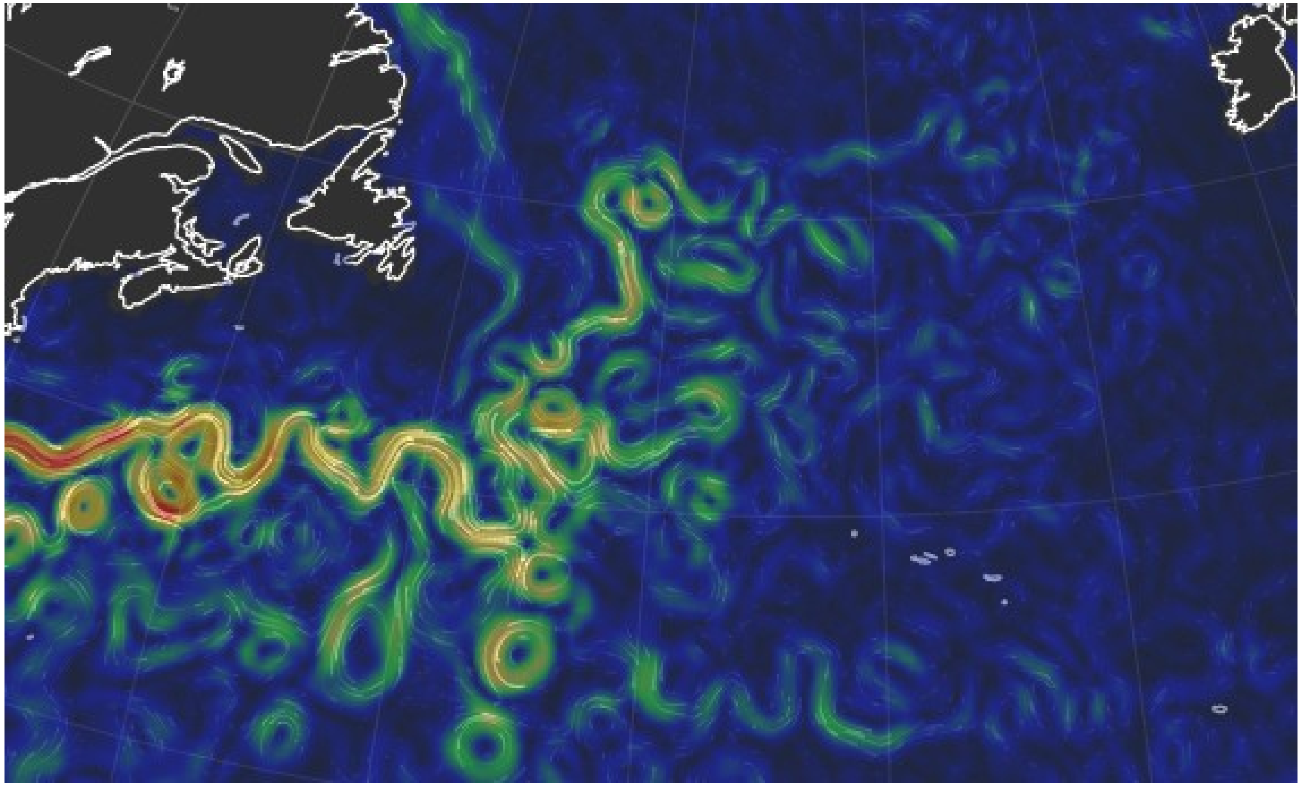
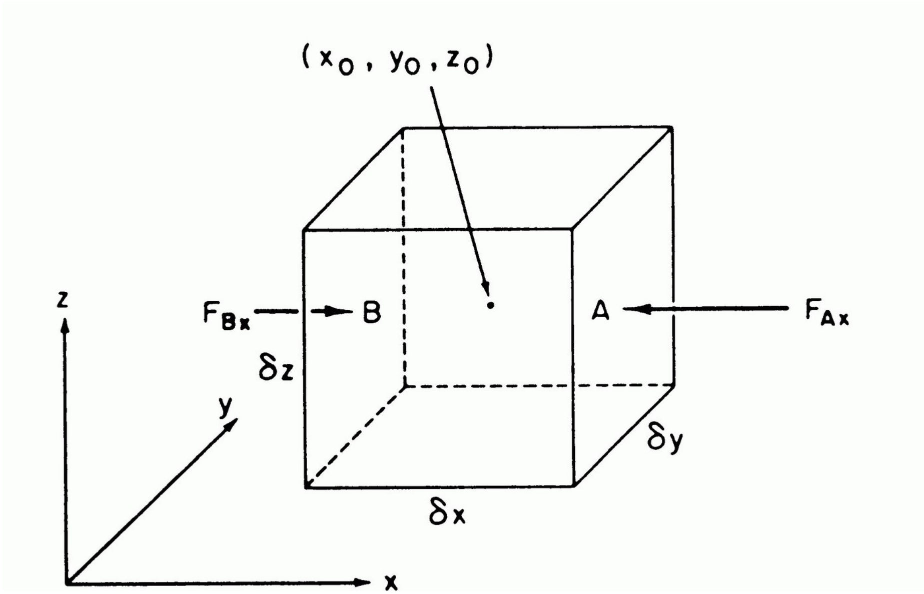
2023-2024 Fall



Equation of motion: forces and scale analysis.

1. Pressure gradient force



$$\frac{\vec{F}}{m} = -\frac{1}{\rho}\vec{\nabla} p,$$

ρ - air density, p – pressure.

2. Gravitational force

M – mass of the Earth, *m* - mass of the elementary volume of air, , gravitational force:

$$\frac{\vec{F}_g}{m} \equiv \vec{g}^{*} = -\frac{GM}{r^2}\left(\frac{\vec{r}}{r}\right) .$$

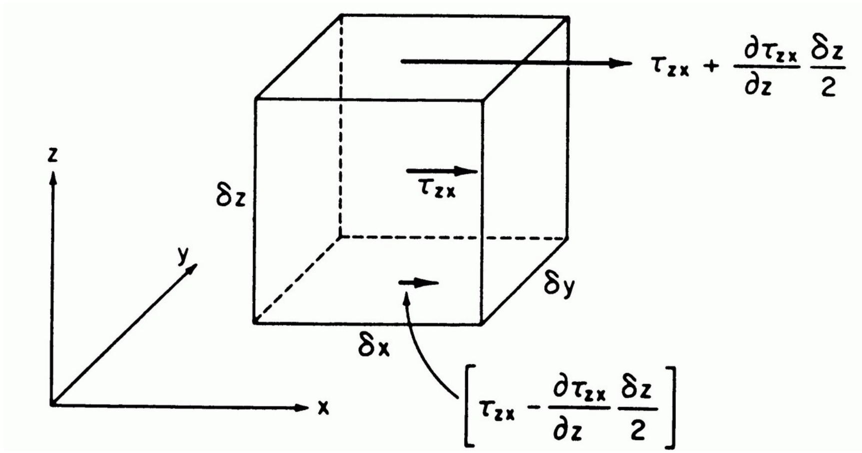
In dynamic meteorlogy and oceanology reference frame is attached to the planetary surface. When *a* is the mean radius of the Earth (mean sea level), neglecting small deflections of the planetary shape from spherical, we may introduce vertical coordinate *z* and $r = a + z$. Then we may introduce:

$$\vec{g}^{*} = \frac{\vec{g}_0^{*}}{(1 + z/a)^2} ,$$

where $\vec{g}_0^{*} = -(GM/a^2)\left(\frac{\vec{r}}{r}\right)$ is gravitational force at the sea level. In meteorology and oceanology z **<<** a , and with a good approximation $\vec{g}^{*} = \vec{g}_0^{*}$.

3. Friction and viscosity.

We know that in many cases friction between the atmosphere and the earth surface cannot be neglected. A kind of friction (viscosity) results from molecular structure of air and water.. Consider elementary volume of air $\delta x \delta y \delta z$:



When the shear stress in direction acting in the middle of the volume is , then the shear stress at the upper side of the volume can be written as:

$$\tau_{zx} + \frac{\partial \tau_{zx}}{\partial z}\frac{\delta z}{2} \quad ,$$

while at the bottom side it is:

$$-\left[\tau_{zx} - \frac{\partial \tau_{zx}}{\partial z}\frac{\delta z}{2}\right] \quad .$$

Total force acting at the volume through its top and bottom is:

$$\left(\tau_{zx} + \frac{\partial \tau_{zx}}{\partial z}\frac{\delta z}{2}\right)\delta y\,\delta x - \left(\tau_{zx} - \frac{\partial \tau_{zx}}{\partial z}\frac{\delta z}{2}\right)\delta y\,\delta x \quad .$$

Thus, friction per unit mass of fluid due to vertical gradient of - velocity component can be written as:

$$\frac{1}{\rho}\frac{\partial \tau_{zx}}{\partial z} = \frac{1}{\rho}\frac{\partial}{\partial z}\left(\mu \frac{\partial u}{\partial z}\right).$$

(since $\delta x \delta y \delta z = \delta V$ and $\frac{M}{\delta V} = \rho \quad \rightarrow \frac{\delta V}{M} = \frac{1}{\rho}$).

For $\mu = const$ the above can be written as:

$$\frac{1}{\rho}\frac{\partial \tau_{zx}}{\partial z} = \nu \frac{\partial^2 u}{\partial z^2} \quad ,$$

where $\nu = \mu / \rho$ is kinematic viscosity, for air $\nu = 1.46 \times 10^{-5} m^2 s^{-1}$.

Considering all three directions we may write:

$$F_{rx} = \nu \left[\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} \right]$$

$$F_{ry} = \nu \left[\frac{\partial^2 \upsilon}{\partial x^2} + \frac{\partial^2 \upsilon}{\partial y^2} + \frac{\partial^2 \upsilon}{\partial z^2} \right]$$

$$F_{rz} = \nu \left[\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2} \right]$$

For the atmosphere, at heights below 100 km kinematic viscosity is so small that can be neglected except for a few cm thick layer close to the surface. The same is true for water.
The effective friction in the atmosphere and ocean results from bulk momentum transport by turbulence and convection, not (directly) by molecular viscosity. ,

From the practical perspective often (not always!!!) eddy viscosity can be introduced to describe friction in turbulent environment. Unfortunately, eddy viscosity is property of particular flow, not of the fluid.

4. Apparent forces: centrifugal force.

Consider air parcel of mass m, rotating along circle of radius r with the angular velocity ω. Consider external reference frame. In order to calculate acceleration

we take velocity difference $\delta \vec{V}$, during time δt in which parcel changes direction by the angle of $\delta\theta$.

$\delta\theta$ is also an angle between $\vec{V}$ and $\vec{V} + \delta\vec{V}$, and $\delta\vec{V}$ is just $\left|\delta\vec{V}\right| = \left|\vec{V}\right| \delta\theta$.

In the limit $\delta t \to 0$, $\delta\vec{V}$ is directed towards axis of rotation:

$$\frac{d\vec{V}}{dt} = \left|\vec{V}\right| \frac{d\theta}{dt} \left(-\frac{\vec{r}}{r} \right),$$

but $\left|\vec{V}\right| = \omega r$ and $d\theta / dt = \omega$, so $\frac{d\vec{V}}{dt} = -\omega^2 \vec{r}$: acceleration is directed towards axis of rotation. In the rotating reference frame the apparent force – centrifugal force is needed to explain parcel motion.

In case of the Earth, rotating with angular velocity Ω ($\Omega = 7.292 \times 10^{-5}$ rad s^{-1}) and distance of a given location from the axis of rotation equal $\vec{R}$,

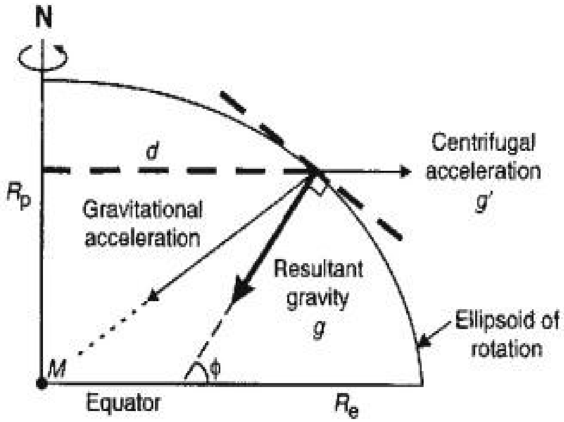
the apparent force acting on the parcel of air is equal to $m * \Omega^2 \vec{R}$.

This force, from the point of view of the observer at the earth's surface can be added to the gravitational force resulting in ….

effective gravitational force - gravity.

4a. effective gravitational force – gravity.

Acceleration due to this force is described as : $\vec{g} \equiv \vec{g}^{\star} + \Omega^2 \vec{R}$.



Gravitational force is directed toward the center of the Earth, while centrifugal force is directed outside from the axis of rotation. In effect, everywhere between the Poles and the Equator gravity does not point toward the center of the earth and the shape of the Earth resembles more ellipsoid than a ball. On the other hand, gravity is always perpendicular to the Earth's surface and deflection from a spherical shape is small: difference between polar and equatorial radii is ~21 km.

Gravity acceleration can be described in terms of potential of certain function

$$\vec{\nabla}\Phi = -\vec{g}$$

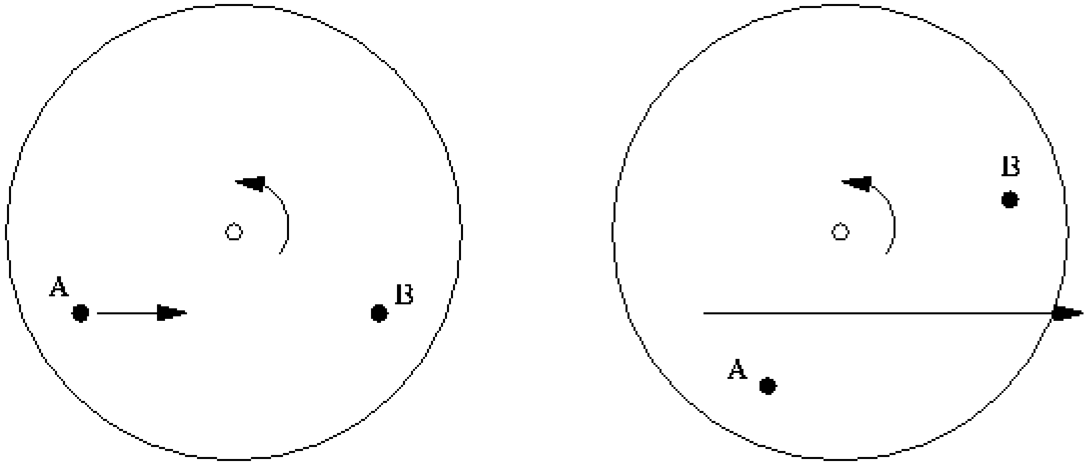
called GEOPOTENTIAL.

Since $\vec{g} = -g\vec{k}$, where $g \equiv \left|\vec{g}\right|$, then $\Phi = \Phi(z)$ and $d\Phi / dz = g$.

For value of geopotential equal to zero at zero height, geopotential $\Phi(z)$ at height z is equal to the work needed to lift unit mass from the mean sea level to the height z

$$\Phi = \int_0^z g dz \ .$$

5. Coriolis force.



Object is propelled from point A on a rotating turntable towards point B. It travels in a straight line but before it reaches B, the turntable has rotated and the object passes to the right of B. To an observer at A the trajectory of the object appears to have been deflected to the right by an imaginary force - the Coriolis force.

Consider air parcel of unit mass moving towards east. Effective rotation of a sum of the rotation of the Earth and motion with respect to the surface of the Earth. When

Ω is the Earth's angular velocity, $\vec{R}$ describes distance from the axis of rotation and u is the eastwards velocity component relative to the surface then:

$$\left(\Omega + \frac{u}{R}\right)^2 \vec{R} = \Omega^2 \vec{R} + \frac{2\Omega u \vec{R}}{R} + \frac{u^2 \vec{R}}{R^2} \quad .$$

$$\left(\Omega + \frac{u}{R}\right)^2 \vec{R} = \Omega^2 \vec{R} + \frac{2\Omega u \vec{R}}{R} + \frac{u^2 \vec{R}}{R^2}$$

First term in the RHS of the equation is the centrifugal acceleration due to Earth's rotation (part of gravity force), second term $2\Omega u\left(\vec{R}/R\right)$ represents Coriolis acceleration, third term, centripetal acceleration, for typical atmospheric motions is negligible, since

$$|u| << \Omega R \quad .$$

Notice that Coriolis acceleration due to west-east motion with respect to the surface of the planet can be decomposed into north-east :

$$\left(\frac{d\upsilon}{dt}\right)_{Co} = -2\Omega u \sin\phi$$,and vertical : $$\left(\frac{dw}{dt}\right)_{Co} = 2\Omega u \cos\phi$$

components.

Here u, υ, w denote eastward, northward and upward velocity components, ϕ is latitude and index Co notes Coriolis effect.

Formally Coriolis force (acceleration) can be introduced in the following way.

We know that total derivative of any vector in the rotating frame is:

$$\frac{D_a \mathbf{A}}{Dt} = \frac{D\mathbf{A}}{Dt} + \mathbf{\Omega} \times \mathbf{A}$$

and in inertial reference frame Newton's second law of motion may be written symbolically as:

$$\frac{D_a \mathbf{U}_a}{Dt} = \sum \mathbf{F}$$

where index "a" stays for absolute.
In order to transform this expression to rotating coordinates, one must first find a relationship between $\mathbf{U_a}$ and the velocity relative to the rotating system **U:**

$$\frac{D_a \mathbf{r}}{Dt} = \frac{D\mathbf{r}}{Dt} + \mathbf{\Omega} \times \mathbf{r}$$

$$D_a \mathbf{r}/Dt \equiv \mathbf{U}_a \text{ and } D\mathbf{r}/Dt \equiv \mathbf{U}$$

which gives:

$$\mathbf{U}_a = \mathbf{U} + \mathbf{\Omega} \times \mathbf{r}$$

Taking total derivative one obtains:

$$\frac{D_a \mathbf{U}_a}{Dt} = \frac{D\mathbf{U}_a}{Dt} + \boldsymbol{\Omega} \times \mathbf{U}_a$$

Substituting from $\mathbf{U_a}$ into the right-hand side one gets:

$$\begin{aligned}\frac{D_a \mathbf{U}_a}{Dt} &= \frac{D}{Dt}(\mathbf{U} + \boldsymbol{\Omega} \times \mathbf{r}) + \boldsymbol{\Omega} \times (\mathbf{U} + \boldsymbol{\Omega} \times \mathbf{r}) \\ &= \frac{D\mathbf{U}}{Dt} + 2\boldsymbol{\Omega} \times \mathbf{U} - \Omega^2 \mathbf{R}\end{aligned}$$

where Ω is constant and **R** is a vector perpendicular to the axis of rotation, with magnitude equal to the distance to the axis of rotation. We used identity:

$$\boldsymbol{\Omega} \times (\boldsymbol{\Omega} \times \mathbf{r}) = \boldsymbol{\Omega} \times (\boldsymbol{\Omega} \times \mathbf{R}) = -\Omega^2 \mathbf{R}$$

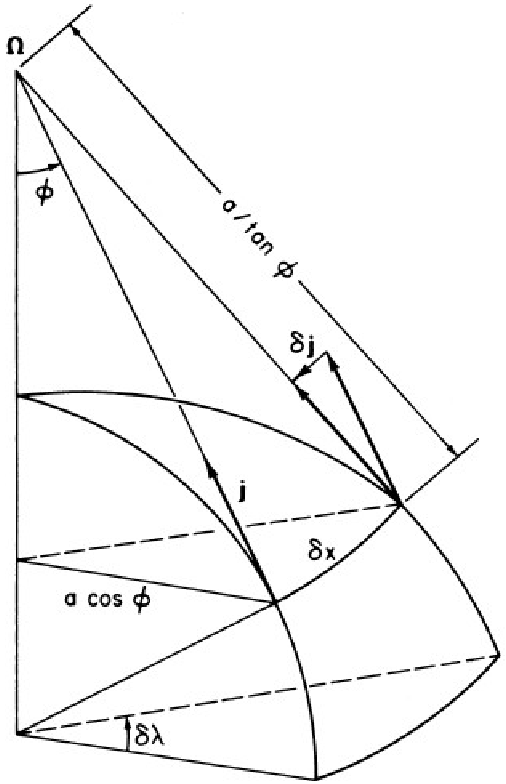
If we assume that the only real forces acting on the atmosphere are the pressure gradient force, gravitation, and friction, we can rewrite Newton's second law with the aid of the above as:

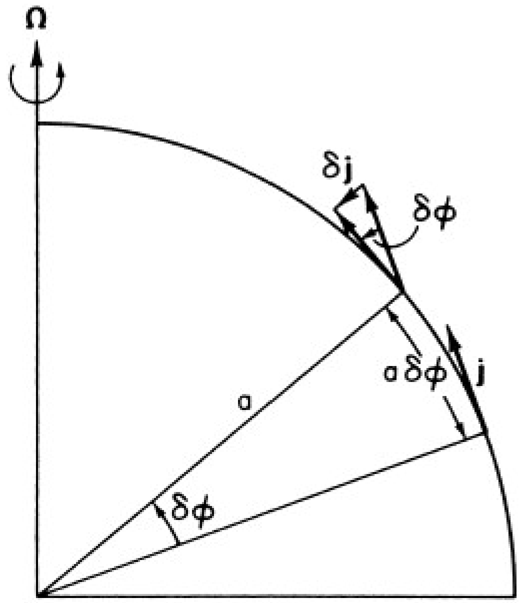
$$\frac{d\vec{U}}{dt} = -2\vec{\Omega} \times \vec{U} - \frac{1}{\rho}\vec{\nabla} p + \vec{g} + \vec{F}_t \quad ,$$

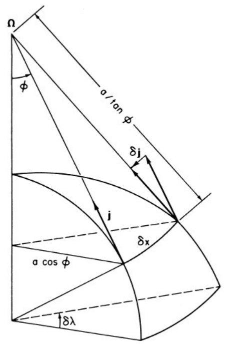
here $\vec{F}_t$ is friction and we combined gravitation and centrifugal forces into gravity .

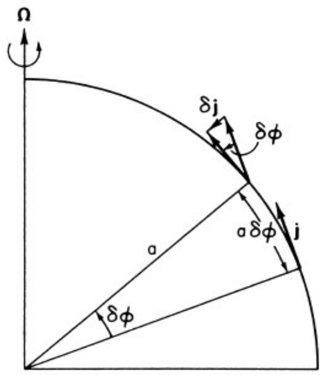
5. Equation of motion.

The above form of the equation of motion is widely used in geophysical fluid dynamics. It is often written in the Cartesian reference frame attached to the Earths surface. Axes are along (λ, ϕ, z), where λ is longitude, ϕ is latitude, and z is vertical.









In such reference frame components are:

$$\frac{du}{dt} - \frac{uv\tan\phi}{a} + \frac{uw}{a} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + 2\Omega v\sin\phi - 2\Omega w\cos\phi + F_{tx}$$

$$\frac{dv}{dt} + \frac{u^2\tan\phi}{a} + \frac{vw}{a} = -\frac{1}{\rho}\frac{\partial p}{\partial y} - 2\Omega u\sin\phi + F_{ty}$$

$$\frac{dw}{dt} - \frac{u^2 + v^2}{a} = -\frac{1}{\rho}\frac{\partial p}{\partial z} - g + 2\Omega u\cos\phi + F_{tz},$$

6. Scale analysis: synoptic motions in mid-latitudes.

We perform scaling the equations of motion in order to determine whether some terms in the equations are negligible for motions of particular concern. Elimination of terms on scaling considerations not only has the advantage of simplifying the mathematics, but as shown later the elimination of small terms in some cases has the very important property of completely eliminating or filtering an unwanted type of motion.

The complete equations of motion describe all types and scales of geophysical flows. Sound waves, for example, are a perfectly valid class of solutions to these equations. However, sound waves are of negligible importance in dynamical meteorology or oceanology. Therefore, it will be a distinct advantage if, as turns out to be true, we can neglect the terms that lead to the production of sound waves and filter out this unwanted class of motions.

In order to simplify equations for synoptic scale motions, we define the following characteristic scales of the field variables based on observed values for midlatitude synoptic systems.

$U \sim 10$ m s^{-1} – horizontal velocity scale

$W \sim 1$ cm s^{-1} – vertical velocity scale

$L \sim 10^6$ m – length scale - 1000 km

$D \sim 10^4$ m – depth scale - 10 km

$\delta P / \rho \sim 10^3$ m^{2}s^{-2} – pressure fluctuations in horizontal

$L/U \sim 10^5$ s – time scale

Horizontal pressure fluctuations δP are normalized by density ρ , to have dcale valid through troposphere depth, since both : δP and ρ decrease (exponentially) with height. Additionally, $\delta P / \rho$ is in the units of geopotential.

Other important scales

$f = 2\,\Omega \sin(\varphi) \approx 1.6 * 10^{-4} \sim 10^{-4}$ s^{-1} – Coriolis parameter at midlatitudes

$a = 6400[km] \sim 10^5$ m – radius of the Earth

L/U is a time to travel distance L with velocity U.

The resulting material derivative of such motions scales as $d/dt \sim U/L$.

Scale analysis of horizontal components of the momentum equation:

	A	B	C	D	E	F	G
X:	$\frac{du}{dt}$	$-2\Omega v \sin\phi$	$+2\Omega w \cos\phi$	$+\frac{uw}{a}$	$-\frac{uv\tan\phi}{a}$	$= -\frac{1}{\rho}\frac{\partial p}{\partial x}$	$+F_{rx}$
Y:	$\frac{dv}{dt}$	$+2\Omega u \sin\phi$		$+\frac{vw}{a}$	$+\frac{u^2\tan\phi}{a}$	$= -\frac{1}{\rho}\frac{\partial p}{\partial y}$	$+F_{ry}$
S:	$\frac{U^2}{L}$	$f_0 U$	$f_0 W$	$\frac{UW}{a}$	$\frac{U^2}{a}$	$\frac{\delta P}{\rho L}$	$\frac{\nu U}{D^2}$
W:	10^{-4}	10^{-3}	10^{-6}	10^{-8}	10^{-5}	10^{-3}	10^{-12}

Geostrophic approximation

first approximation of the above scale analysis results in:

$$fv \approx \frac{1}{\rho}\frac{\partial p}{\partial x} \quad , \quad fu \approx -\frac{1}{\rho}\frac{\partial p}{\partial y} \quad ,$$

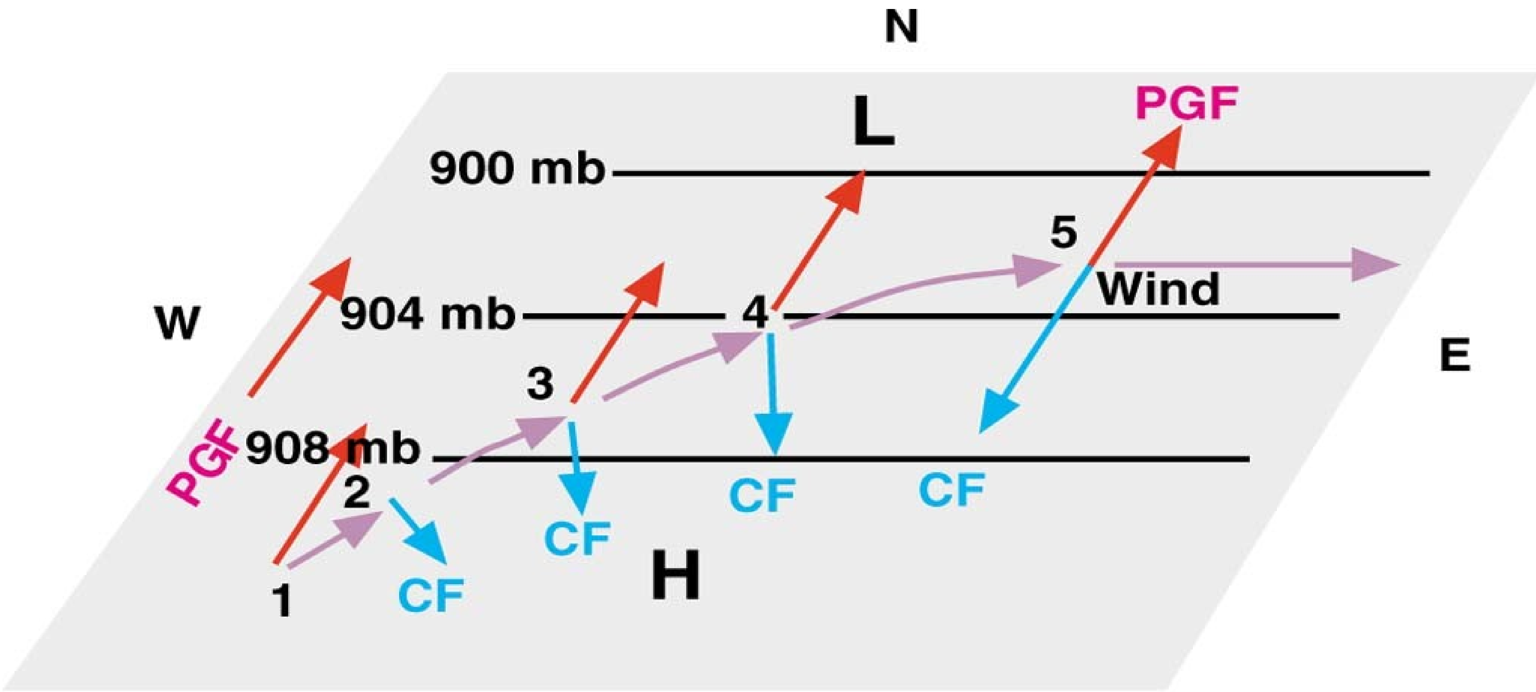
where $f \equiv 2\Omega \sin\phi$ is Coriolis parameter.

Flow resulting from balance between Coriolis and pressure gradient forces

$\vec{V}_g \equiv \vec{i}\, u_g + \vec{j}\, v_g$ is called geostrophic wind defined as:

$$\vec{V}_g \equiv \vec{k} \times \frac{1}{\rho f}\vec{\nabla} p \quad .$$

In mid-latitudes geostrophic wind differs from the actual wind for 10-15%.


N
L
PGF
900 mb
5
Wind
W
904 mb
4
E
3
PGF
908 mb
2
CF
CF
CF
CF
H
1
CF
S

Prognostic approximation, Rossby' number.

The next approximation of equations of motion takes the form:

$$\frac{du}{dt} = fv - \frac{1}{\rho}\frac{\partial p}{\partial x} = f\left(v - v_g\right),$$

$$\frac{dv}{dt} = -fu - \frac{1}{\rho}\frac{\partial p}{\partial y} = -f\left(u - u_g\right)$$

This approximation is called "quasi-geostrophic" or "prognostic"

Ratio of the acceleration in the above equation to Coriolis is:

$$\frac{U^2/L}{f_0 U} = \frac{U}{f_o L} \equiv R_0 \quad ,$$

R_0 Rossby number.

Scale analysis of the vertical component:

Because pressure decreases by about an order of magnitude from the ground to the tropopause, the vertical pressure gradient may be scaled by P_0 /H , where P_0 is the surface pressure and H is the depth of the troposphere.

Table 2.2 *Scale Analysis of the Vertical Momentum Equation*

z - Eq.	Dw/Dt	$-2\Omega u \cos\phi$	$-(u^2+v^2)/a$	$= -\rho^{-1}\partial p/\partial z$	$-g$	$+F_{rz}$
Scales	UW/L	$f_0 U$	U^2/a	$P_0/(\rho H)$	g	$\nu W H^{-2}$
m s^{-2}	10^{-7}	10^{-3}	10^{-5}	10	10	10^{-15}

The scaling indicates that to a high degree of accuracy the pressure field is in **hydrostatic equilibrium**; that is, the pressure at any point is simply equal to the weight of a unit cross-section column of air above that point.

The above analysis of the vertical momentum equation is, however, somewhat misleading. It is not sufficient to show merely that the vertical acceleration is small compared to g. Because only that part of the pressure field that varies horizontally is directly coupled to the horizontal velocity field, it is actually necessary to show that the horizontally varying pressure component is itself in hydrostatic equilibrium with the horizontally varying density field.

Let's define standard pressure $p_0(z)$, which is the horizontally averaged pressure at each height, and a corresponding standard density $\rho_0(z)$, which are in exact hydrostatic balance:

$$\frac{1}{\rho_0}\frac{dp_0}{dz} \equiv -g$$

Let's write:

$$p(x, y, z, t) = p_0(z) + p'(x, y, z, t)$$
$$\rho(x, y, z, t) = \rho_0(z) + \rho'(x, y, z, t)$$

where p' and ρ' are deviations from the standard values of pressure and density. For an atmosphere at rest, p' and ρ' would thus be zero. Then, assuming $\rho'/\rho_0 \ll 1$ results in:

$$(\rho_0 + \rho')^{-1} \cong \rho_0^{-1}\,(1 - \rho'/\rho_0)$$

and

$$-\frac{1}{\rho}\frac{\partial p}{\partial z} - g = -\frac{1}{(\rho_0 + \rho')}\frac{\partial}{\partial z}\left(p_0 + p'\right) - g$$
$$\approx \frac{1}{\rho_0}\left[\frac{\rho'}{\rho_0}\frac{dp_0}{dz} - \frac{\partial p'}{\partial z}\right] = -\frac{1}{\rho_0}\left[\rho' g + \frac{\partial p'}{\partial z}\right]$$

For synoptic scale motions

$$\frac{1}{\rho_0}\frac{\partial p'}{\partial z} \sim \left[\frac{\delta P}{\rho_0 H}\right] \sim 10^{-1}\mathrm{m\ s^{-2}}, \quad \frac{\rho' g}{\rho_0} \sim 10^{-1}\mathrm{m\ s^{-2}}$$

which can be compared with terms in Table 2.
To a very good approximation the perturbation pressure field is in hydrostatic equilibrium with the perturbation density field and:

$$\frac{\partial p'}{\partial z} + \rho' g = 0$$

Therefore, for synoptic scale motions, vertical accelerations are negligible.

This leads to the approximation in which the vertical velocity cannot be determined from the vertical momentum equation, it has to be deduced indirectly!

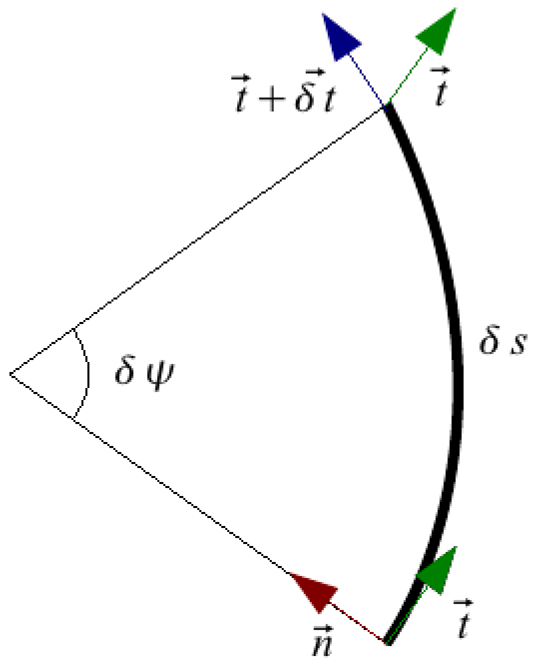
However, fact that pressure is related to density with hydrostatic relation:

$$\frac{1}{\rho_0}\frac{dp_0}{dz} \equiv -g$$

allows usage of pressure as vertical coordinate!

Balanced flows.

Natural Coordinates



The natural coordinate system is defined by the orthogonal set of unit vectors **t**, **n**, and **k**.
Unit vector **t** is oriented parallel to the horizontal velocity at each point;
unit vector **n** is normal to the horizontal velocity and is oriented so that it is positive to the left of the flow direction;
unit vector **k** is directed vertically upward.
In this system the horizontal velocity may be written **V** = V **t** where V , the horizontal speed, is a non-negative scalar defined by V ≡ Ds/Dt,
s(x, y, t) is the distance along the curve followed by a parcel moving in the horizontal plane.
Then:

$$\frac{D\mathbf{V}}{Dt} = \frac{D(V\,\mathbf{t})}{Dt} = \mathbf{t}\frac{DV}{Dt} + V\frac{D\mathbf{t}}{Dt}$$

and

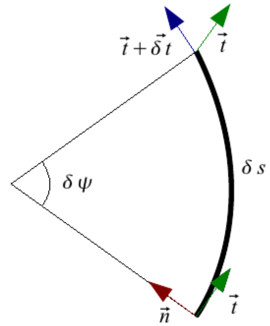
$$\delta\psi = \frac{\delta s}{|R|} = \frac{|\delta\mathbf{t}|}{|\mathbf{t}|} = |\delta\mathbf{t}|$$

R is the radius of curvature following the parcel motion, by convention, R is positive when the center of curvature is in the positive n direction. Thus, for R > 0, the air turn toward the left following the motion, and for R < 0 toward the right.

We may calculate:

$$\frac{D\mathbf{t}}{Dt} = \frac{D\mathbf{t}}{Ds}\frac{Ds}{Dt} = \frac{\mathbf{n}}{R}V$$

$$\frac{D\mathbf{V}}{Dt} = \mathbf{t}\frac{DV}{Dt} + \mathbf{n}\frac{V^2}{R}$$



The Coriolis force always acts normal to the direction of motion:

$$-f\,\mathbf{k}\times\mathbf{V} = -f\,V\,\mathbf{n}$$

and pressure gradient can be expresses as:

$$-\nabla_p\Phi = -\left(\mathbf{t}\frac{\partial\Phi}{\partial s} + \mathbf{n}\frac{\partial\Phi}{\partial n}\right)$$

in natural coordinates.

The horizontal momentum equation can be expanded into components :

$$\frac{DV}{Dt} = -\frac{\partial \Phi}{\partial s}$$

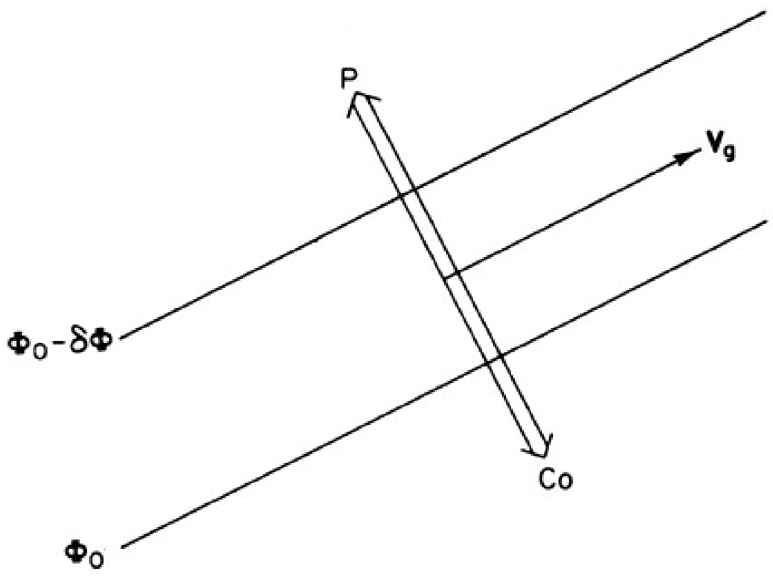
$$\frac{V^2}{R} + fV = -\frac{\partial \Phi}{\partial n}$$

For motion parallel to the geopotential height contours, $\partial\Phi/\partial s = 0$ and the speed is constant. When the geopotential gradient (normal to the direction of motion) is constant the above implies constant radius trajectory at given latitude.
In that case the flow can be classified into several simple categories.

1. Geostrophic Flow

Flow in a straight line ($R \to \pm\infty$) parallel to height contours.

$$fV_g = -\partial\Phi/\partial n$$



2. Inertial Flow
   For uniform geopotential on an isobaric surface the horizontal pressure gradient vanishes, and a balance between Coriolis force and centrifugal force exists:

$$V^2/R + fV = 0$$

The above gives:

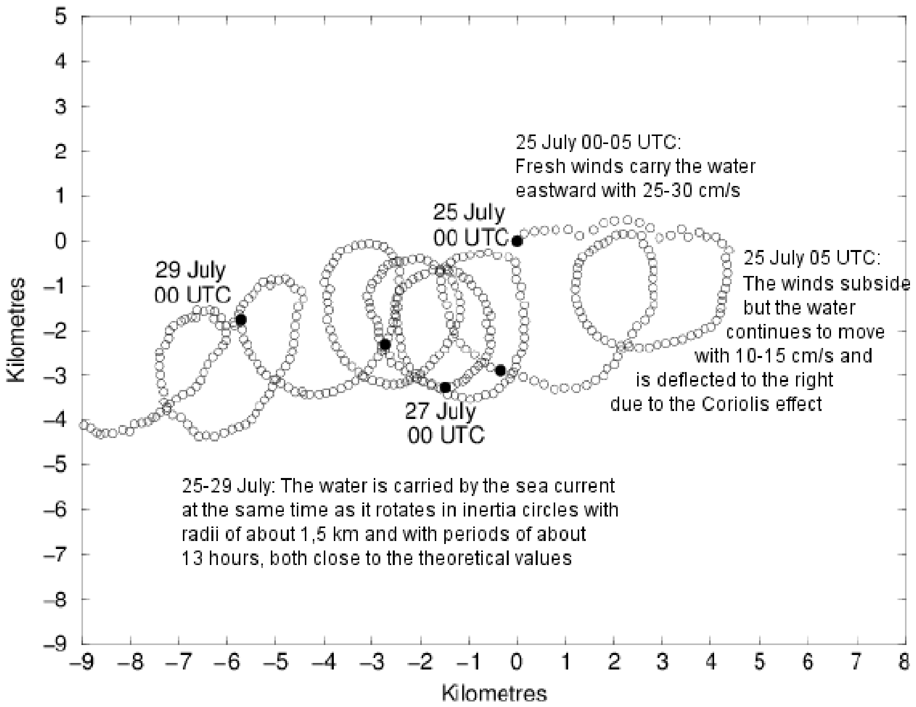
$$R = -V/f$$

Fluid air parcels follow circular paths in an anticyclonic sense. The period of this oscillation is:

$$P = \left|\frac{2\pi R}{V}\right| = \frac{2\pi}{|f|} = \frac{\frac{1}{2} day}{|\sin\phi|}$$

Because both the Coriolis force and the centrifugal force due to the relative motion are caused by inertia of the fluid, this type of motion is traditionally referred to as an inertial oscillation, and the circle of radius |R| is called the inertia circle.

Notice that inertial oscillations are important for the ocean surface!!!!!



A drifting buoy set in motion by strong westerly winds in the Baltic Sea in July 1969. When the wind has decreased the uppermost water layers of the oceans tend to follow approximately inertia circles due to the Coriolis effect. This is reflected in the motions of drifting buoys. In the case there are steady ocean currents the trajectories will become cycloides. The inertia circles are not eddies; a set of buoys close to each other would be co-moving, rather than revolve around each other..

Consider inertial oscillation visualizer at:
http://oceanmotion.org/html/resources/coriolis.htm

4. Cyclostrophic Flow
   If the horizontal scale of a flow is small enough, the Coriolis force may be neglected in compared to the pressure gradient force and the centrifugal force and:

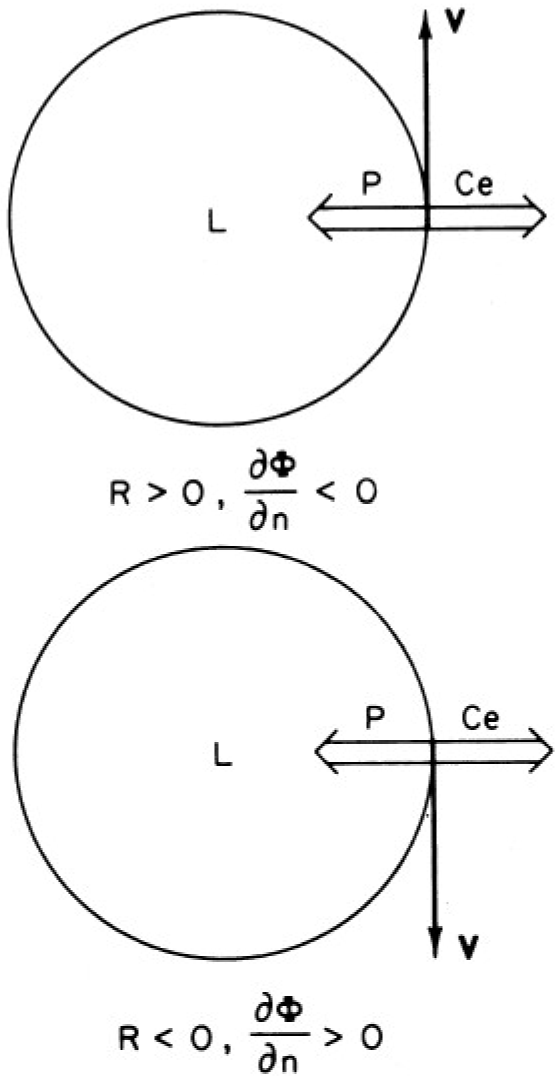
$$\frac{V^2}{R} = -\frac{\partial \Phi}{\partial n}$$

$$V = \left(-R\frac{\partial \Phi}{\partial n}\right)^{1/2}$$

This flow can be either cyclonic or anticyclonic.
The cyclostrophic balance approximation is valid provided that the ratio of the centrifugal force to the Coriolis force $V/(fR)$ large. This ratio is equivalent to the Rossby number
Consider a typical tornado of the tangential velocity of 30 ms^{-1} at a distance of 300 m from the center of the vortex. Assuming that $f = 10^{-4}\ s^{-1}$, the Rossby number is :

$$R_o = V/|f R| \approx 10^3.$$

Fig. 3.4 Force balance in cyclostrophic flow: *P* designates the pressure gradient and *Ce* the centrifugal force.



4. **The Gradient Wind Approximation**
   Horizontal frictionless flow that is parallel to the height contours so that the tangential acceleration vanishes (DV /Dt = 0) is called gradient flow. Gradient flow is a three-way balance among the Coriolis force, the centrifugal force, and the horizontal pressure gradient force:

Solving the balance one gets:

$$V = -\frac{fR}{2} \pm \left(\frac{f^2R^2}{4} - R\frac{\partial \Phi}{\partial n}\right)^{1/2}$$
$$= -\frac{fR}{2} \pm \left(\frac{f^2R^2}{4} + fRV_g\right)^{1/2}$$

Not all the mathematically possible roots of the above correspond to physically possible solutions, as it is required that V be real and non-negative. The various roots are classified according to the signs of R and ∂ /∂n in order to isolate the physically meaningful solutions.

Table 3.1 *Classification of Roots of the Gradient Wind Equation in the Northern Hemisphere*

Sign $\partial\Phi/\partial n$	$R > 0$	$R < 0$
Positive ($V_g < 0$)	Positive root:[a] unphysical	Positive root: antibaric flow (anomalous low)
	Negative root: unphysical	Negative root: unphysical
Negative ($V_g > 0$)	Positive root: cyclonic flow (regular low)	Positive root: ($V > -fR/2$): anticyclonic flow (anomalous high)
	Negative root: unphysical	Negative root: ($V < -fR/2$): anticyclonic flow (regular high)

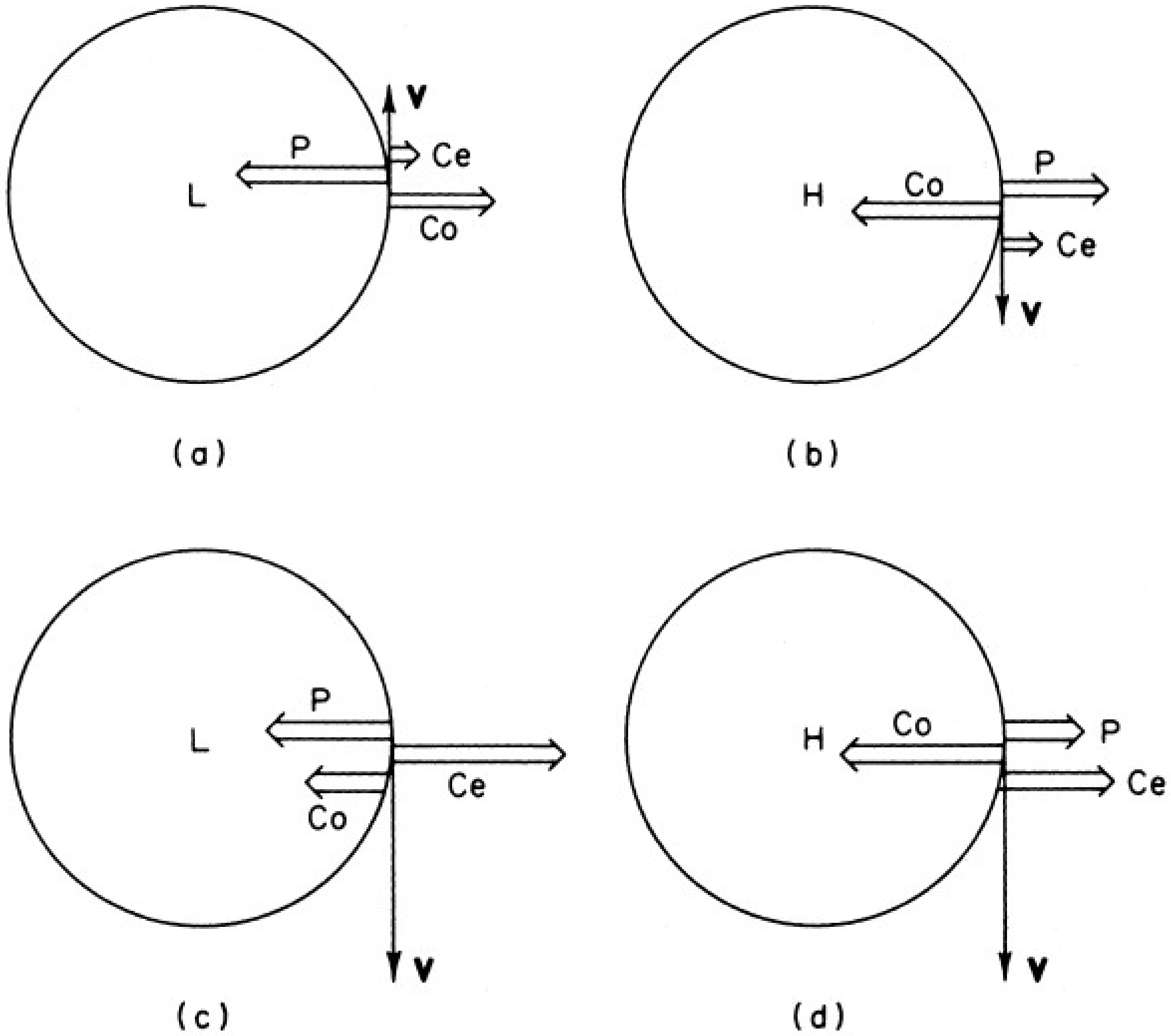


Fig. 3.5 Force balances in the Northern Hemisphere for the four types of gradient flow: (a) regular low (b) regular high (c) anomalous low (d) anomalous high.