Dynamics of the Atmosphere and the Ocean

Lecture 10

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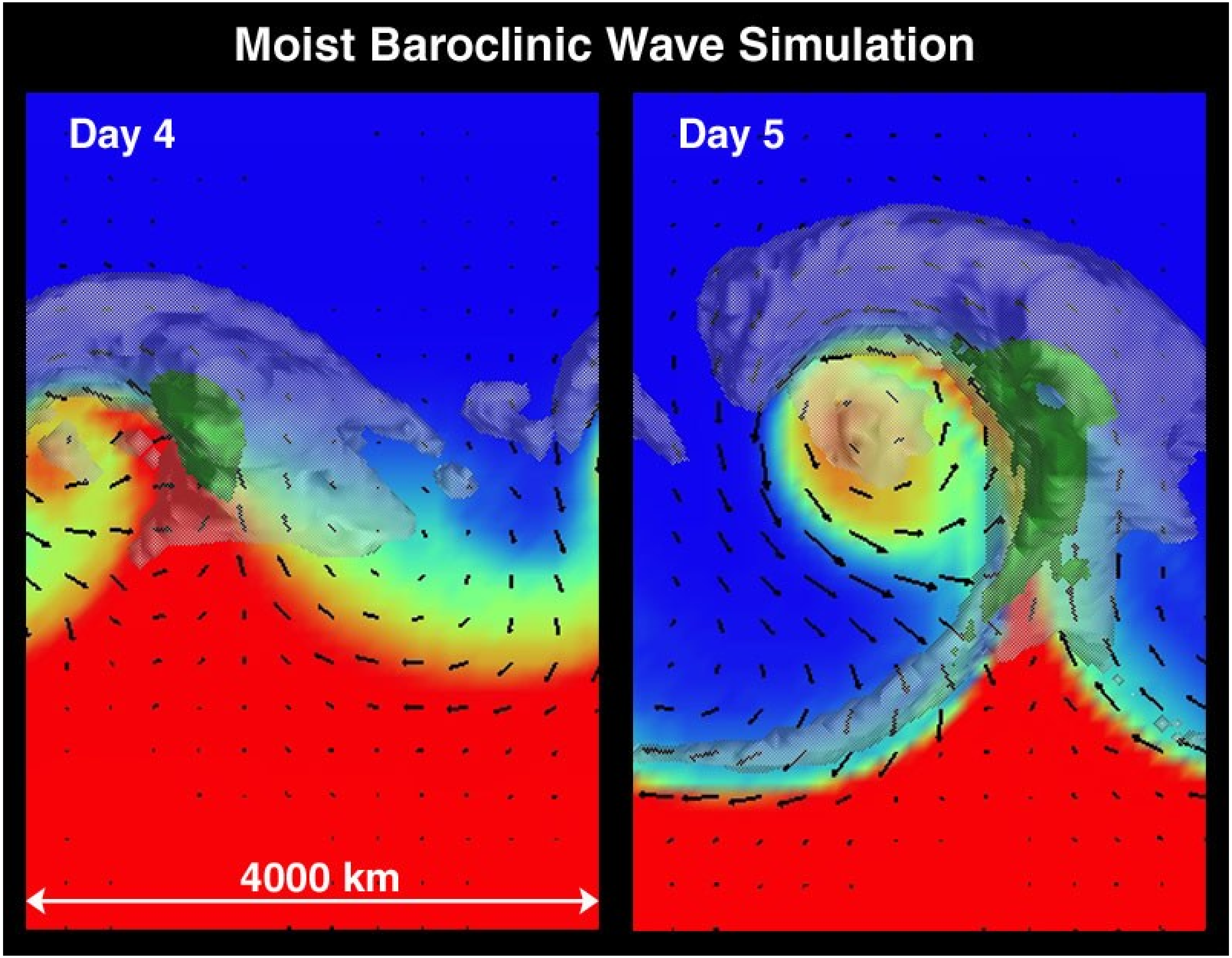
2023-2024 Fall



QUASI-GEOSTROPHIC PREDICTION

The evolution of the geostrophic circulation can actually be determined without explicitly determining the distribution of ω. Defining the geopotential tendency $\chi \equiv \partial\Phi/\partial t$, and recalling that the order of partial differentiation may be reversed, the geostrophic vorticity equation can be expressed as

$$\frac{1}{f_0}\nabla^2\chi = -\mathbf{V}_g \bullet \nabla\left(\frac{1}{f_0}\nabla^2\Phi + f\right) + f_0\frac{\partial\omega}{\partial p}$$

An analogous equation dependent on these two variables can be obtained from the thermodynamic energy equation by multiplying through by f_0/σ and differentiating with respect to p. Using the definition of χ , the result can be expressed as

$$\frac{\partial}{\partial p}\left(\frac{f_0}{\sigma}\frac{\partial\chi}{\partial p}\right) = -\frac{\partial}{\partial p}\left[\frac{f_0}{\sigma}\mathbf{V}_g \bullet \nabla\left(\frac{\partial\Phi}{\partial p}\right)\right] - f_0\frac{\partial\omega}{\partial p} - f_0\frac{\partial}{\partial p}\left(\frac{\kappa J}{\sigma p}\right)$$

The left side of can be interpreted as the local rate of change of a normalized static stability anomaly:

$$\frac{\partial}{\partial p}\left(\frac{f_0}{\sigma}\frac{\partial\chi}{\partial p}\right) = -Rf_0\frac{\partial}{\partial p}\left(\frac{1}{\sigma p}\frac{\partial T}{\partial t}\right) = -f_0\frac{\partial}{\partial p}\left(\frac{1}{S_p}\frac{\partial T}{\partial t}\right) \approx -\frac{\partial}{\partial t}\left(\frac{f_0}{S_p}\frac{\partial T}{\partial p}\right)$$

$$\frac{\partial \Phi}{\partial p} = -\alpha = -RT/p$$

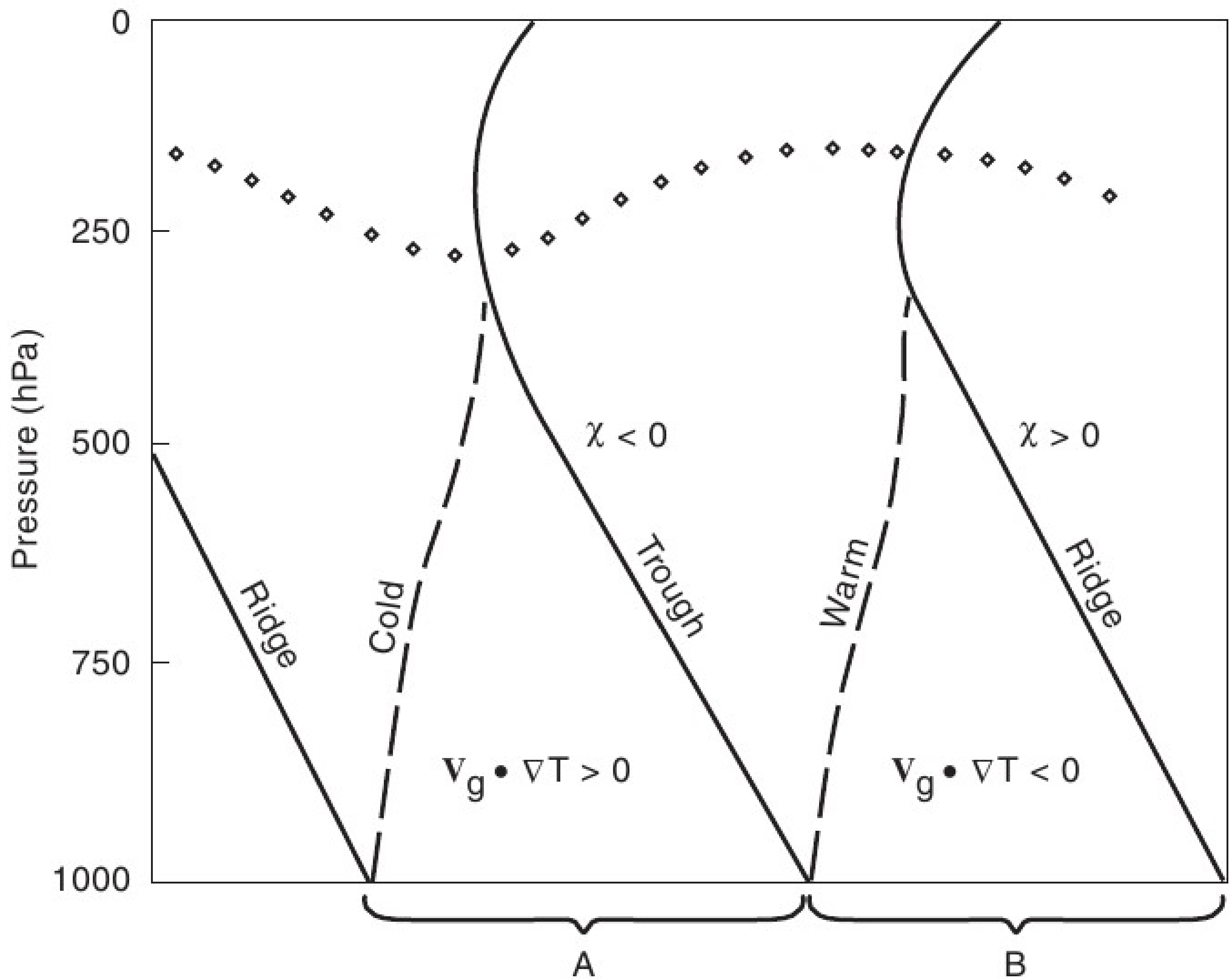
Geopotential Tendency

If for simplicity we set J = 0 and eliminate ω between the equations on the previous slide to obtain an equation that determines the local rate of change of geopotential in terms of the three-dimensional distribution of the geopotential field:

$$\underbrace{\left[\nabla^2 + \frac{\partial}{\partial p}\left(\frac{f_0^2}{\sigma}\frac{\partial}{\partial p}\right)\right]\chi}_{\mathrm{A}} = -\underbrace{f_0 \mathbf{V}_g \cdot \nabla\left(\frac{1}{f_0}\nabla^2\Phi + f\right)}_{\mathrm{B}}$$

$$- \underbrace{\frac{\partial}{\partial p}\left[-\frac{f_0^2}{\sigma}\mathbf{V}_g \cdot \nabla\left(-\frac{\partial \Phi}{\partial p}\right)\right]}_{\mathrm{C}}$$

This equation is often referred to as the geopotential tendency equation. It provides a relationship between the local geopotential tendency (term A) and the distributions of vorticity advection (term B) and thickness advection (term C). If the distribution of is known at a given time, then terms B and C may be regarded as known forcing functions, and yje above is a linear partial differential equation in the unknown χ.



East–west section through a developing synoptic disturbance showing the relationship of temperature advection to the upper level height tendencies. A and B designate, respectively, regions of cold advection and warm advection in the lower troposphere.

The quasi-geostrophic vorticity equation can be obtained from the x and y components of the quasi-geostrophic momentum equation:

$$\frac{D_g u_g}{Dt} - f_0 v_a - \beta y v_g = 0$$

$$\frac{D_g v_g}{Dt} + f_0 u_a + \beta y u_g = 0$$

Taking spatial derivatives and using the fact that the divergence of the geostrophic wind vanishes, yields the vorticity equation:

$$\frac{D_g \zeta_g}{Dt} = -f_0 \left(\frac{\partial u_a}{\partial x} + \frac{\partial v_a}{\partial y} \right) - \beta v_g$$

$$\mathrm{D}_g f / Dt = \mathbf{V}_g \bullet \boldsymbol{\nabla} f = \beta v_g$$

$$\frac{\partial \zeta_g}{\partial t} = -\mathbf{V}_g \bullet \boldsymbol{\nabla} \left(\zeta_g + f \right) + f_0 \frac{\partial \omega}{\partial p}$$

which states that the local rate of change of geostrophic vorticity is given by the sum of the advection of the absolute vorticity by the geostrophic wind plus the concentration or dilution of vorticity by stretching or shrinking of fluid columns (the divergence effect).

The Quasi-Geostrophic Vorticity Equation

The vertical component of vorticity can be approximated geostrophically:

$$f_0 v_g = \frac{\partial \Phi}{\partial x}, \qquad f_0 u_g = -\frac{\partial \Phi}{\partial y}$$

$$\zeta_g = \mathbf{k} \cdot \boldsymbol{\nabla} \times \mathbf{V}_g$$

$$\zeta_g = \frac{\partial v_g}{\partial x} - \frac{\partial u_g}{\partial y} = \frac{1}{f_0}\left(\frac{\partial^2 \Phi}{\partial x^2} + \frac{\partial^2 \Phi}{\partial y^2}\right) = \frac{1}{f_0}\boldsymbol{\nabla}^2 \Phi$$

The above equation can be used to determine ζ_g (x, y) from a known field Φ(x, y). Alternatively, it can be solved by inverting the Laplacian operator to determine from a known distribution of ζ_g , provided that suitable conditions on Φ are specified on the boundaries of the region in question.

This invertibility is one reason why vorticity is such a useful forecast diagnostic; if the evolution of the vorticity can be predicted, then inversion of the equation yields the evolution of the geopotential field, from which it is possible to determine the geostrophic wind and temperature distributions.

Since the Laplacian of a function tends to be a maximum where the function itself is a minimum, positive vorticity implies low values of geopotential and vice versa.

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This equation is often referred to as the geopotential tendency equation. It provides a relationship between the local geopotential tendency (term A) and the distributions of vorticity advection (term B) and thickness advection (term C). If the distribution of is known at a given time, then terms B and C may be regarded as known forcing functions, and yje above is a linear partial differential equation in the unknown χ.

The Quasi-Geostrophic Potential Vorticity Equation

The geopotential tendency equation:

$$\underbrace{\left[\nabla^2 + \frac{\partial}{\partial p}\left(\frac{f_0^2}{\sigma}\frac{\partial}{\partial p}\right)\right]}_{\text{A}}\chi = \underbrace{-f_0\mathbf{V}_g \bullet \nabla\left(\frac{1}{f_0}\nabla^2\Phi + f\right)}_{\text{B}}$$

$$\underbrace{-\frac{\partial}{\partial p}\left[-\frac{f_0^2}{\sigma}\mathbf{V}_g \bullet \nabla\left(-\frac{\partial\Phi}{\partial p}\right)\right]}_{\text{C}}$$

is useful for physical motivation of processes leading to geopotential changes (and hence upperlevel troughing and ridging), as the tendency χ is related to vorticity and temperature advection. However, this form of the equation actually conceals its true character as a conservation equation for a field commonly referred to as quasi-geostrophic potential vorticity. To put the tendency equation in conservation form we simplify the right-hand side by using the chain rule of differentiation to write term C as:

$$-\mathbf{V}_g \bullet \nabla\frac{\partial}{\partial p}\left(\frac{f_0^2}{\sigma}\frac{\partial\Phi}{\partial p}\right) - \frac{f_0^2}{\sigma}\frac{\partial\mathbf{V}_g}{\partial p} \bullet \nabla\frac{\partial\Phi}{\partial p}$$

However, from the thermal wind relation, $f_0\ \partial V_g/\partial p = k \times \nabla(\partial\Phi/\partial p)$, so that $\partial V_g/\partial p$ is perpendicular to $\nabla(\partial\Phi/\partial p)$, and the second term vanishes. The first term (which is proportional to the geostrophic advection of the normalized static stability anomaly) can be combined with term B

Recalling the definition of χ and dividing through by f_0 , the tendency equation can be expressed simply in the form of a conservation equation:

$$\left(\frac{\partial}{\partial t} + \mathbf{V}_g \bullet \boldsymbol{\nabla}\right) q = \frac{D_g q}{Dt} = 0$$

In the above q is the quasi-geostrophic potential vorticity defined by

$$q \equiv \left[\frac{1}{f_0}\boldsymbol{\nabla}^2\Phi + f + \frac{\partial}{\partial p}\left(\frac{f_0}{\sigma}\frac{\partial \Phi}{\partial p}\right)\right]$$

The three parts of above, reading from left to right, are the geostrophic relative vorticity, the planetary vorticity, and the stretching vorticity. As a parcel moves about in the atmosphere, the geostrophic relative vorticity, the planetary vorticity, and the stretching vorticity terms may each change. However, their sum is conserved following the geostrophic motion.

According to the definition, quasi-geostrophic potential vorticity tends to be proportional to minus the geopotential. A local increase in q is associated with trough development, whereas a decrease in q is associated with ridge development. Because q is a conserved quantity following the geostrophic motion, we can diagnose the tendency purely from the geostrophic advection of q.

The Traditional Omega Equation

Because ζ_g and V_g are both defined in terms of (x, y, p, t), the vorticity equation can be used to diagnose the ω field provided that the fields of both Φ and ∂Φ /∂t are known.

To obtain the omega equation we take the horizontal Laplacian of the energy equation:

$$\left(\frac{\partial}{\partial t} + \mathbf{V}_g \bullet \nabla\right)\left(-\frac{\partial \Phi}{\partial p}\right) - \sigma\omega = \frac{\kappa J}{p}$$

to yield

$$\nabla^2 \frac{\partial \chi}{\partial p} = -\nabla^2\left[\mathbf{V}_g \bullet \nabla\left(\frac{\partial \Phi}{\partial p}\right)\right] - \sigma\nabla^2\omega - \frac{\kappa}{p}\nabla^2 J$$

Next, the geostrophic vorticity equation used to derive the tendency equation:

$$\frac{1}{f_0}\nabla^2\chi = -\mathbf{V}_g \bullet \nabla\left(\frac{1}{f_0}\nabla^2\Phi + f\right) + f_0\frac{\partial\omega}{\partial p}$$

should be differentiated with respect to pressure providing:

$$\frac{\partial}{\partial p}\left(\nabla^2\chi\right) = -f_0\frac{\partial}{\partial p}\left[\mathbf{V}_g \bullet \nabla\left(\frac{1}{f_0}\nabla^2\Phi + f\right)\right] + f_0^2\frac{\partial^2\omega}{\partial p^2}$$

Because the order of the operators on the left-hand side in the above equations may be reversed, the result of subtracting is to eliminate χ . After rearrangement of terms, we obtain the traditional omega equation:

$$\underbrace{\left(\nabla^2 + \frac{f_0^2}{\sigma}\frac{\partial^2}{\partial p^2}\right)}_{\mathrm{A}}\omega = \frac{f_0}{\sigma}\frac{\partial}{\partial p}\underbrace{\left[\mathbf{V}_g \bullet \nabla\left(\frac{1}{f_0}\nabla^2\Phi + f\right)\right]}_{\mathrm{B}}$$

$$+ \underbrace{\frac{1}{\sigma}\nabla^2\left[\mathbf{V}_g \bullet \nabla\left(-\frac{\partial\Phi}{\partial p}\right)\right]}_{\mathrm{C}} - \underbrace{\frac{\kappa}{\sigma p}\nabla^2 J}_{\mathrm{D}}$$

Although terms B and C in the omega equation apparently have clear interpretations as separate physical processes, in practice there is often a significant amount of cancellation between them.
They also are not invariant under a Galilean transformation of the zonal coordinate. That is, adding a constant mean zonal velocity will change the magnitude of each of these terms without changing the net forcing of the vertical motion. For these reasons, an alternative approximate form of the omega equation is often applied in synoptic analyses.

Employing the chain rule of differentiation yields for term B:

$$\frac{f_0}{\sigma}\left[\frac{\partial \mathbf{V}_g}{\partial p} \bullet \nabla\left(\frac{1}{f_0}\nabla^2\Phi + f\right)\right] + \frac{1}{\sigma}\mathbf{V}_g \bullet \nabla\left(\frac{\partial \nabla^2 \Phi}{\partial p}\right)$$

and for term C:

$$-\frac{1}{\sigma}\left[\left(\nabla^2 u_g\right)\frac{\partial}{\partial x}\left(\frac{\partial \Phi}{\partial p}\right) + \left(\nabla^2 v_g\right)\frac{\partial}{\partial y}\left(\frac{\partial \Phi}{\partial p}\right)\right] - \frac{1}{\sigma}\mathbf{V}_g \bullet \nabla\left(\frac{\partial \nabla^2 \Phi}{\partial p}\right)$$

The second terms the above are equal and opposite, whereas the first term in the second formula is generally much smaller than the first term in the first one. Thus, for adiabatic flow, the omega, neglecting the smaller term, results in approximate form of the omega equation:

$$\underbrace{\left(\nabla^2 + \frac{f_0^2}{\sigma}\frac{\partial^2}{\partial p^2}\right)}\omega \approx \frac{f_0}{\sigma}\left[\frac{\partial \mathbf{V}_g}{\partial p} \bullet \nabla\left(\frac{1}{f_0}\nabla^2\Phi + f\right)\right]$$

The above involves only derivatives in space. It is a diagnostic equation for the field of ω in terms of the instantaneous Φ field. The omega equation, unlike the continuity equation, provides a method of estimating ω that does not depend on observations of the ageostrophic wind. However, second order derivatives of observed variables make practical use difficult.

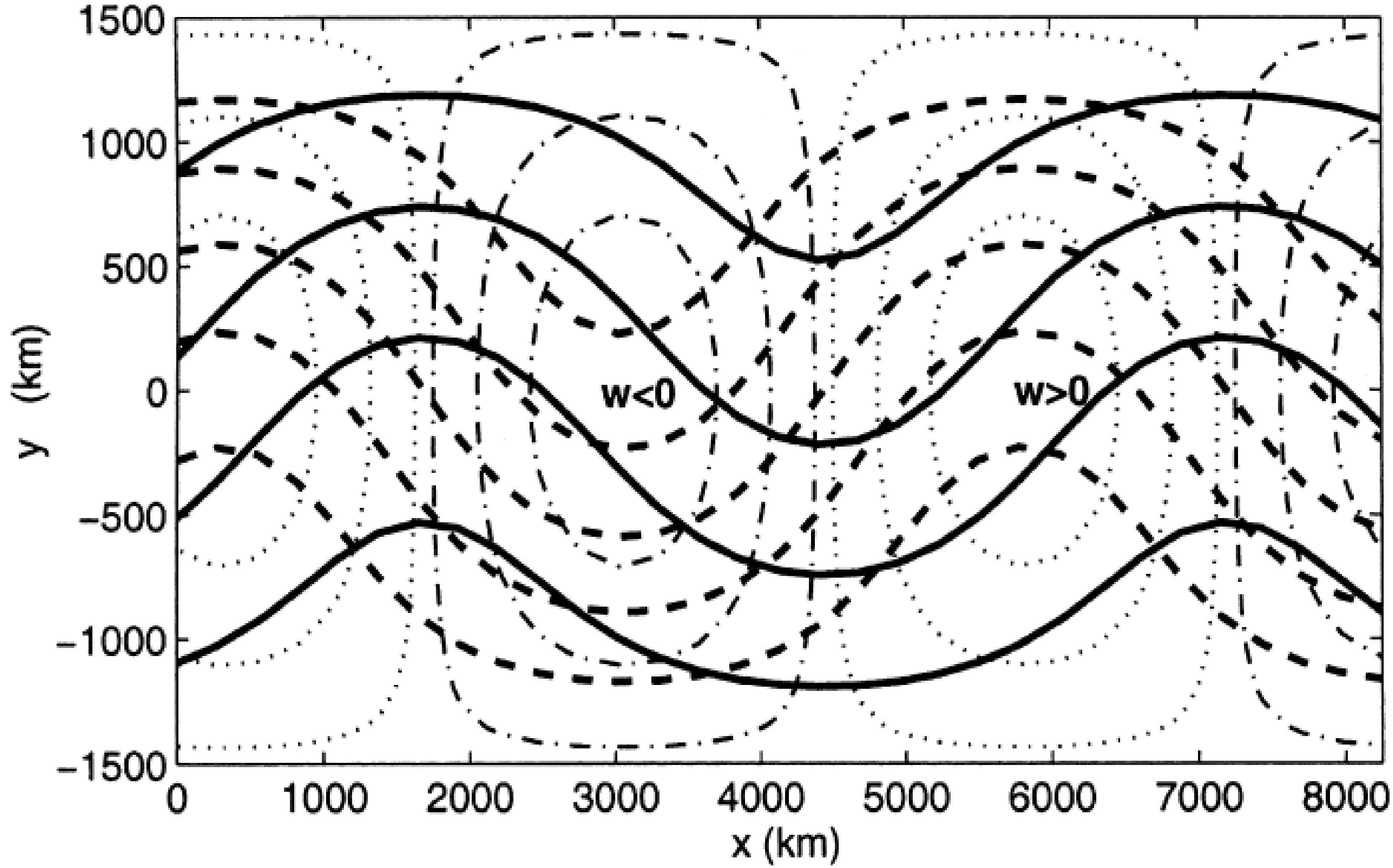
The terms in the omega equation can be physically interpreted by first observing that the differential operator on the left is very similar to the operator in term A of the tendency equation) and hence tends to spread the response to a localized forcing.

Because this forcing tends to be a maximum in the midtroposphere, and ω is required to vanish at the upper and lower boundaries, for qualitative discussion it is permissible to assume that ω has sinusoidal behavior not only in the horizontal, but also in the vertical:

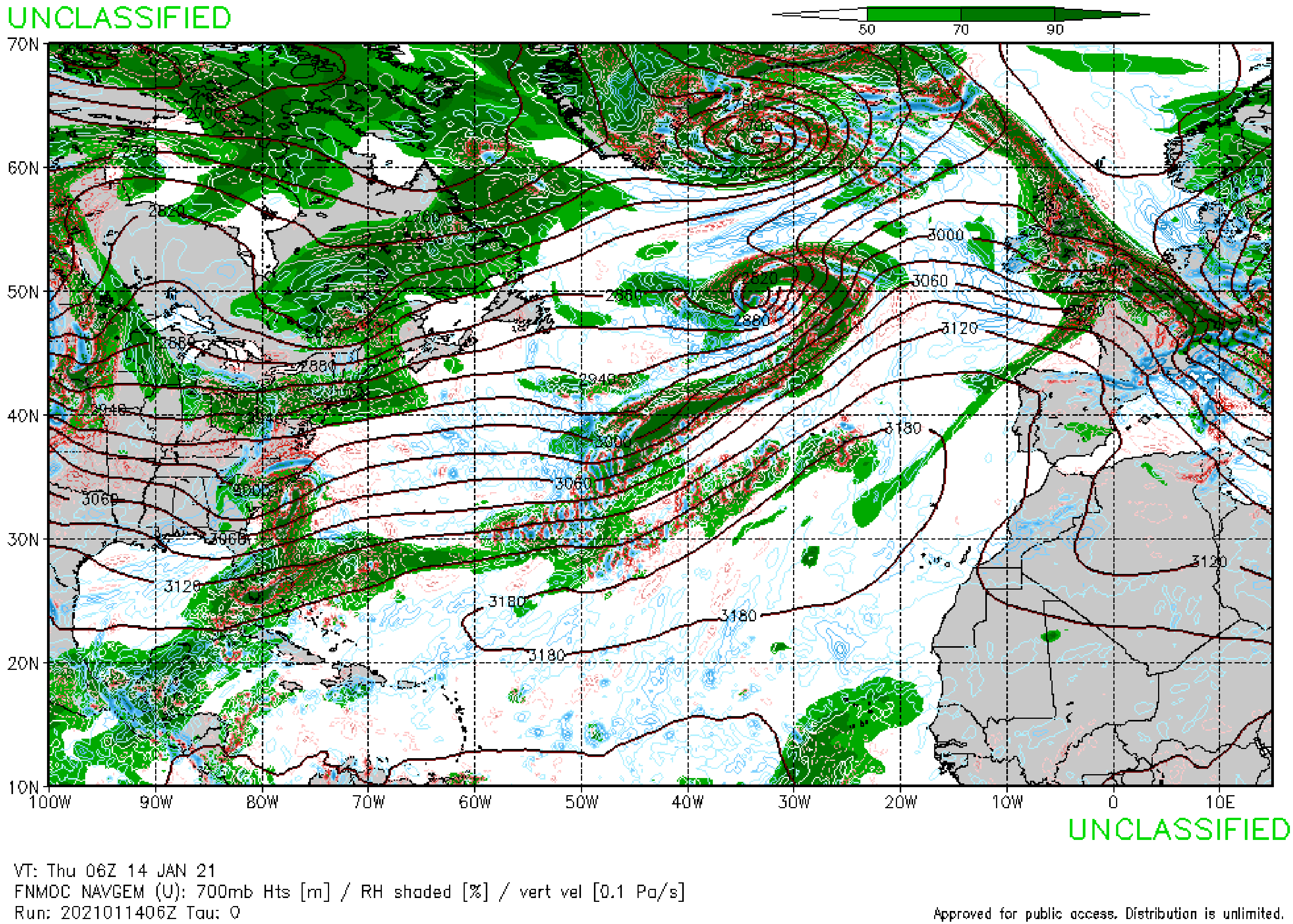
$$\omega = W_0 \sin\left(\pi p / p_0\right) \sin kx \ \sin ly$$

$$\left(\nabla^2 + \frac{f_0^2}{\sigma}\frac{\partial^2}{\partial p^2}\right)\omega \approx -\left[k^2 + l^2 + \frac{1}{\sigma}\left(\frac{f_0 \pi}{p_0}\right)^2\right]\omega$$

Thus, upward motion is forced where the right-hand side of omega equation is positive and downward motion is forced where it is negative.

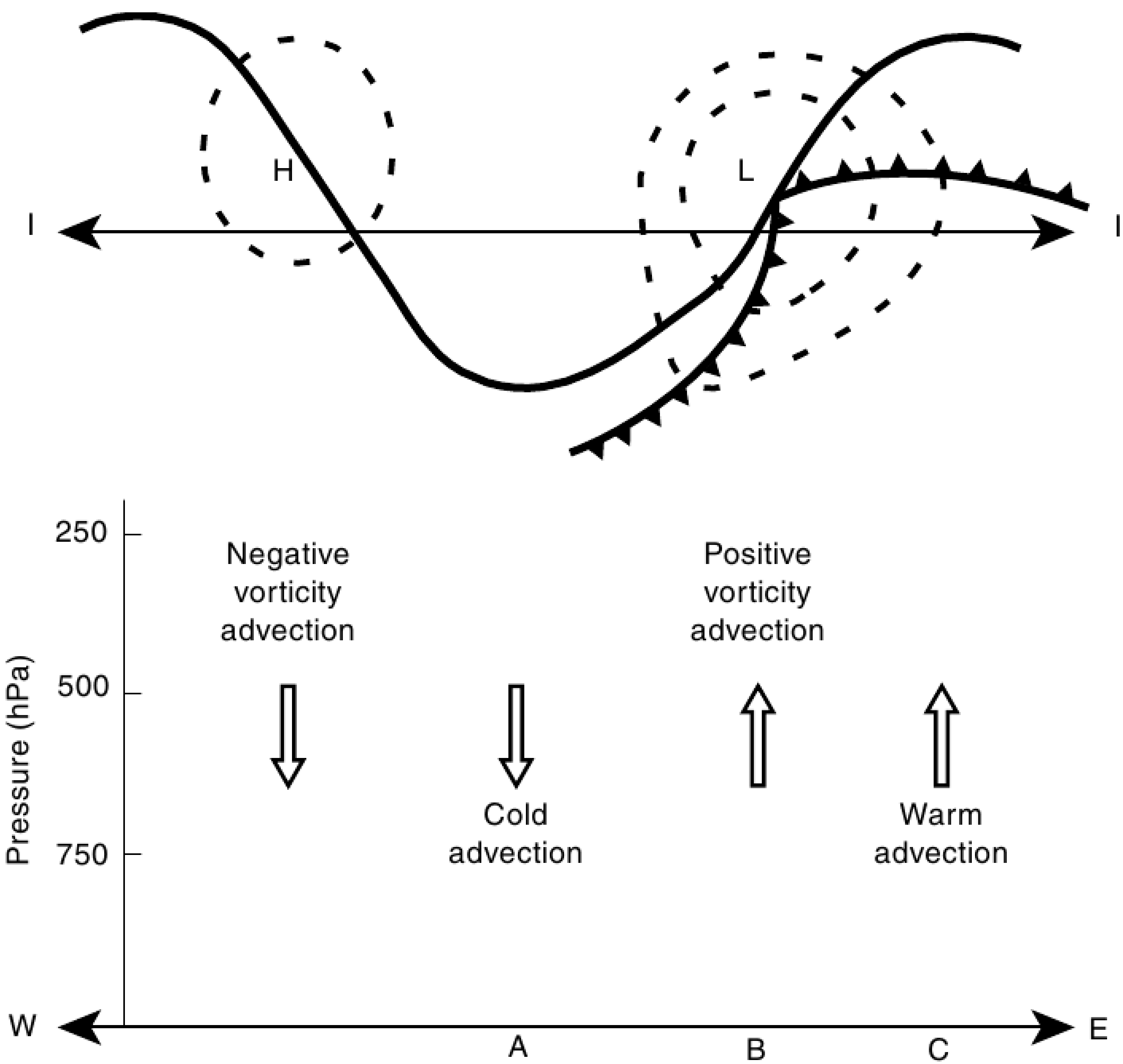


Schematic 500-hPa height contours (solid lines), isotherms (dashed lines), and vertical motion field (w > 0 dash-dot lines, w < 0 dotted lines) for a developing synoptic-scale system. Upward motion occurs where vorticity decreases moving left to right along an isotherm, and downward motion occurs where vorticity decreases moving left to right along an isotherm.



Vertical velocity (omega) on 700hPa over Europe (see color contours).

IDEALIZED MODEL OF A BAROCLINIC DISTURBANCE



Secondary circulation associated with a developing baroclinic wave:

(top) schematic 500-hPa contour (solid line), 1000-hPa contours (dashed lines), and surface fronts;

(bottom) vertical profile through line II' indicating the vertical motion field.

Table 6.1 *Characteristics of a Developing Baroclinic Disturbance*

Physical parameter	A 500-hPa trough	B Surface low	C 500-hPa ridge
$\partial(\delta\Phi)/\partial t$ (500–1000 hPa)	Negative (thickness advection partly canceled by adiabatic warming)	Negative (adiabatic cooling)	Positive (thickness advection partly canceled by adiabatic cooling)
w (500 hPa)	Negative	Positive	Positive
$\partial\Phi/\partial t$ (500 hpa)	Negative (differential thickness advection)	Negative (vortity advection)	Positive (differential thickness advection)
$\partial\zeta_g/\partial t$ (1000 hPa)	Negative (divergence)	Positive (convergence)	Positive (convergence)
$\partial\zeta_g/\partial t$ (500 hPa)	Positive (convergence)	Positive (advection partly canceled by divergence)	Negative (divergence)