Solutions for extra exercises for final exam

1. We express:

$$\bar{u} = U + u',$$

$$h = H + h'.$$

Our equations then look like:

$$\frac{\partial(U + u')}{\partial t} + (U + u') \cdot \nabla(U + u') = -g\nabla(H + h'),$$

$$\frac{\partial(H + h')}{\partial t} + (U + u') \cdot \nabla(H + h') + (H + h')(\nabla \cdot (U + u')) = 0,$$

considering that U and H are constant we then get:

$$\frac{\partial u'}{\partial t} + (U + u') \cdot \nabla u' = -g\nabla h',$$

$$\frac{\partial h'}{\partial t} + (U + u') \cdot \nabla h' + (H + h')(\nabla \cdot u') = 0,$$

and expanding:

$$\frac{\partial u'}{\partial t} + U \cdot \nabla u' + u' \cdot \nabla u' = -g\nabla h',$$

$$\frac{\partial h'}{\partial t} + U \cdot \nabla h' + u' \cdot \nabla h' + H(\nabla \cdot u') + h'(\nabla \cdot u') = 0,$$

we now assume that terms like $x' \cdot \nabla y' = 0$:

$$\frac{\partial u'}{\partial t} + U \cdot \nabla u' = -g\nabla h',$$

$$\frac{\partial h'}{\partial t} + U \cdot \nabla h' + H(\nabla \cdot u') = 0.$$

2. While crossing the mountain barrier, the trajectory of the wind changes. It curves and changes its vorticity. Solution:

   (a) Our initial velocity is $\bar{u} = [u(y), 0]$. Therefore:

$$\zeta_i = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = +\frac{10\text{m/s}}{1000000\text{m}} = +10^{-5}1/\text{s}$$

(b) We consider the conservation of potential vorticity, where everything changes:

$$\zeta_f = \frac{H_i}{H_f}(\zeta_i + f_i - f_f) = \frac{10}{7}(\zeta_i + 2\Omega(\sin(43) - \sin(38))) = 2.81 \cdot 10^{-5} 1/\mathrm{s}$$

(c)

$$\zeta_f = \omega = \frac{v_f}{R},$$

$$R = \frac{v}{\zeta_f} = \frac{2.81 \cdot 10^{-5} 1/\mathrm{s}}{20\mathrm{m/s}} = 7.11 \cdot 10^5 \mathrm{m} = 7.11 \cdot 10^2 \mathrm{km}$$

3.

$$C = 2\pi R U = 2\pi R U_0 \left(\frac{R}{R_0+1}\right)^m = 2\pi U_0 \frac{R^{m+1}}{(R_0+1)^m}$$

$$\zeta = -\frac{\partial U}{\partial n} + \frac{U}{R} = \frac{\partial U}{\partial R} + \frac{U}{R} = m U_0 \frac{R^{m-1}}{(R_0+1)^m} + U_0 \frac{R^{m-1}}{(R_0+1)^m} = (m+1)\frac{U}{R}$$

Cyclostrophic balance:

$$\frac{U^2}{R} = -\frac{\partial \phi}{\partial n} = \frac{\partial \phi}{\partial R} = -\frac{1}{\rho}\frac{\partial p}{\partial R},$$

Integral over R:

$$-\frac{P - P_0}{\rho} = \int_{R_0}^{R} U_0^2 \frac{R^{2n-1}}{(R_0+1)^2 n} \mathrm{d}R = \frac{U_0^2}{(R_0+1)^{2n}} \frac{R^{2n} - R_0^{2n}}{2n},$$

$$P = -\frac{\rho}{2n}\frac{U_0^2}{(R_0+1)^{2n}}(R^{2n} - R_0^{2n}) + P_0$$

4. (a) Calculate the rotation of the equation (1). Then, use the vector identities and the fact that $\nabla \times (\nabla \phi) = 0$, $\nabla \cdot (\nabla \times \bar{\phi}) = 0$ and $\nabla \times (\alpha \nabla \phi) = \nabla \alpha \times \nabla \phi$. Then consider the fact that:

$$\frac{\mathrm{d}}{\mathrm{d}x}\frac{1}{f(x)} = -\frac{1}{f^2(x)}\frac{\mathrm{d}f}{\mathrm{d}x}$$

(b) Inviscid: $\bar{F} = 0$, incompressible: $\nabla \cdot \mathrm{v} = 0$, barotropic: $\rho = \rho(p)$. The first two assumptions simplify the third and the fifth term. The third assumption is basically saying that the angle between the surface of constant pressure and the surface of constant density is zero.

(c) If you put equation (2) into the third term of the equation from the first exercise, and divide by ρ, then you end up with:

$$\frac{1}{\rho}\frac{\mathrm{D}\bar{\omega}}{\mathrm{D}t} = \frac{1}{\rho}(\bar{\omega}\cdot\nabla)\bar{v} + \frac{\bar{\omega}}{\rho^2}\frac{\mathrm{D}\rho}{\mathrm{D}t} + \frac{1}{\rho^3}(\nabla\rho\times\nabla p) + \frac{1}{\rho}\nabla\times\bar{F},$$

then, comparing to the equation we need to prove, we need to show that:

$$\frac{1}{\rho}\frac{\mathrm{D}\bar{\omega}}{\mathrm{D}t} - \frac{\bar{\omega}}{\rho^2}\frac{\mathrm{D}\rho}{\mathrm{D}t} = \frac{\mathrm{D}}{\mathrm{D}t}(\frac{\bar{\omega}}{\rho}).$$

It is true, if you apply a rule for integration of the multiplication of two functions.