Recent progress in the symmetry based theory for near-surface shear flows

Martin Oberlack[1,2]
Sergio Hoyas[3]
Simon Görtz[1,2]
Francisco Alcántara-Ávila[3]
Jonathan Laux[1]
Dario Klingenberg[1,2]
Paul Hollmann[1]

[1] Technische Universität Darmstadt, Germany
[2] Centre for Computational Engineering, TU Darmstadt, Germany
[3] Universitat Politècnica de València, Spain

MASCHINENBAU
We engineer future
FDY

Funding:
MINECO/FEDER RTI2018-102256-B-I00, GVA/FEDER ACIF2018, DFG grant No. OB96/39-1, LRZ SuperMUC project pr92la.

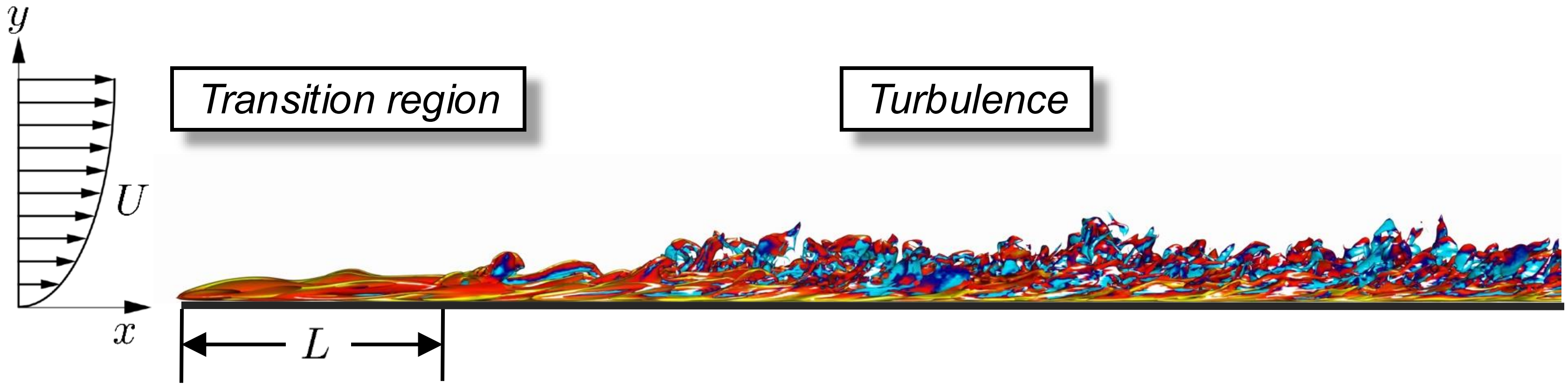
Introduction

Onset of turbulence - transition

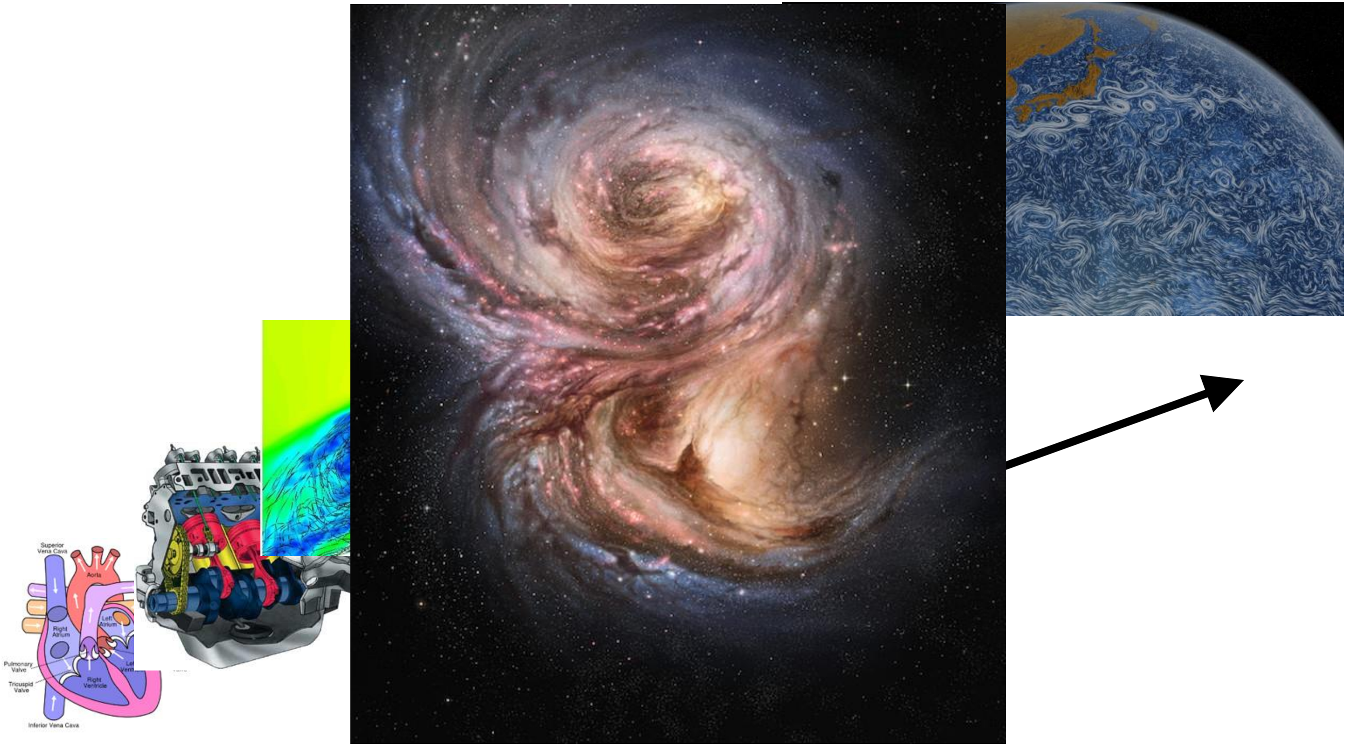
- The onset of turbulence

$$Re = \frac{\textbf{Inertial Forces}}{\textbf{Viscous Forces}} = \frac{\rho U L}{\mu} > Re_{krit}$$

- Boundary layer transition to turbulence



Appearance of turbulence



Model equation – Navier-Stokes

- Navier-Stokes equation – one of the 6 *unsolved* millennium problems of mathematics

$$\frac{\partial \boldsymbol{U}}{\partial t} + (\boldsymbol{U} \cdot \nabla)\,\boldsymbol{U} = -\nabla P + \nu \Delta \boldsymbol{U}$$

$$\nabla \cdot \boldsymbol{U} = 0$$

To-date the “best model for turbulence”!

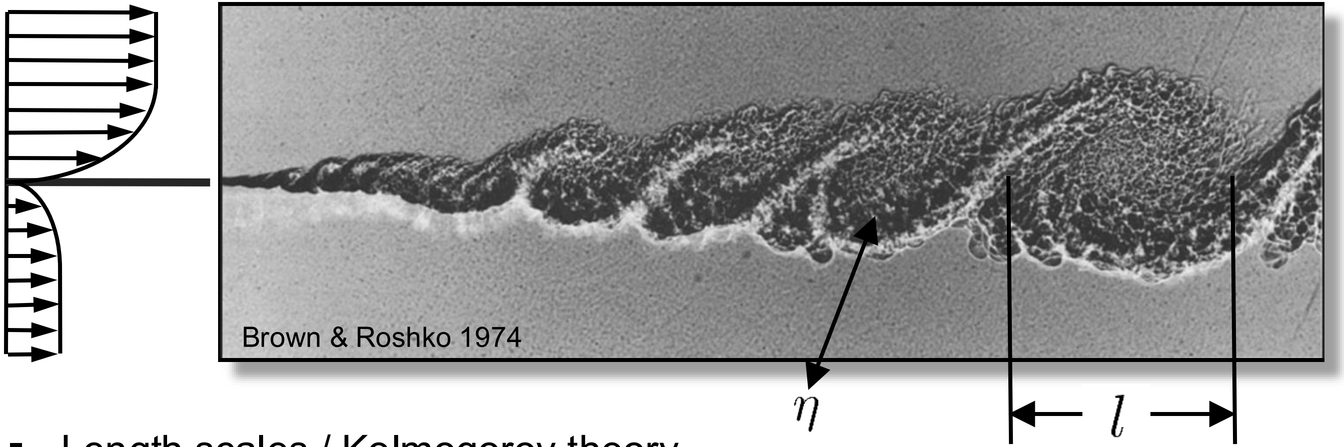
Model equation – Navier-Stokes

- Navier-Stokes equation – one of the 6 *unsolved* millennium problems of mathematics

$$\frac{\partial \boldsymbol{U}}{\partial t} + (\boldsymbol{U} \cdot \nabla) \boldsymbol{U} = - \nabla \frac{1}{4\pi} \int \frac{\partial}{\partial \boldsymbol{x}'} \left(\boldsymbol{u}' \cdot \frac{\partial \boldsymbol{u}'}{\partial \boldsymbol{x}'} \right) \frac{\mathrm{d}\boldsymbol{x}'}{|\boldsymbol{x} - \boldsymbol{x}'|} + \nu \Delta \boldsymbol{U}$$

To-date the "best model for turbulence"!

Length scales in turbulence



- Length scales / Kolmogorov theory
  - Smallest scale: Kolmogorov length scale η
  - Largest scale: Integral length scale l
- Length scale ratio: $$\frac{l}{\eta} = Re^{\frac{3}{4}}$$

Computational cost of turbulence

UNIVERSITAT POLITÈCNICA DE VALÈNCIA
TECHNISCHE UNIVERSITÄT DARMSTADT

- Degrees of freedom (DOF)

$$\#DOF \sim \left(\frac{l}{\eta}\right)^3 = Re^{\frac{9}{4}}$$

- Time scale ratio

$$\frac{T}{\tau_\eta} = Re^{\frac{1}{2}}$$

- Reynolds-scaling of computational cost

$$\boxed{N = \frac{T}{\tau_\eta}\left(\frac{l}{\eta}\right)^3 = Re^{\frac{11}{4}} \approx Re^3}$$

Computational Cost A380

- Chord Reynolds number $Re \approx 10^8 \quad \Rightarrow \quad \boxed{\#DOF \sim 10^{18}}$

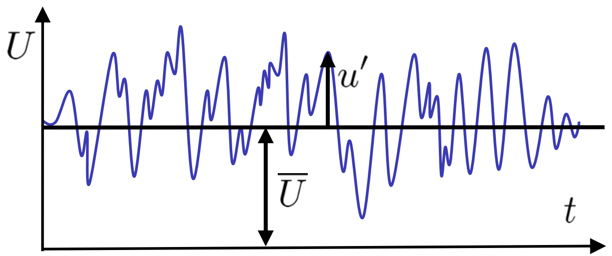
- According to Moor's law / Kryder's law

Full Navier-Stokes simulation for A380 about **4 decades** away!

Turbulence closure problem

UNIVERSITAT POLITÈCNICA DE VALÈNCIA
TECHNISCHE UNIVERSITÄT DARMSTADT

- Statistics of turbulence



- Reynolds decomposition

$$\overline{\boldsymbol{U}} = \lim_{T \to \infty} \frac{1}{T} \int_0^T \boldsymbol{U} \, \mathrm{d}t$$

$$\boldsymbol{U} = \overline{\boldsymbol{U}} + \boldsymbol{u}'$$

$$\frac{\partial \overline{\boldsymbol{U}}}{\partial t} + (\overline{\boldsymbol{U}} \cdot \nabla)\overline{\boldsymbol{U}} = -\nabla \overline{P} + \nu \Delta \overline{\boldsymbol{U}} - \nabla \cdot \boxed{\overline{\boldsymbol{u}' \otimes \boldsymbol{u}'}}$$

Unclosed Reynolds stress tensor

$\Rightarrow$ Empirical closure relations needed or considering all higher moments!

Turbulent scaling laws & log-law

- According to Prandtl / von Karman

$$\frac{\mathrm{d}\bar{U}_1}{\mathrm{d}x_2} = f(u_\tau, x_2, \cancel{\nu})$$

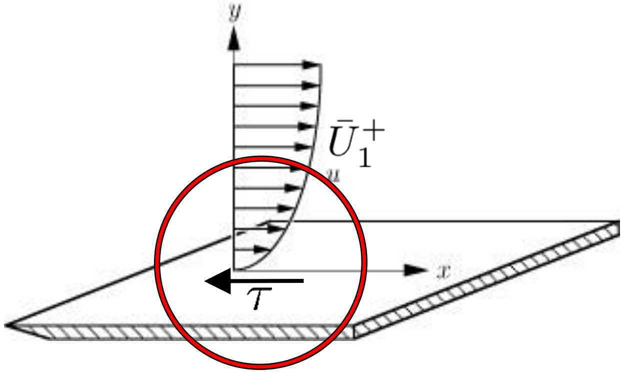
- with wall-friction velocity

$$u_\tau = \sqrt{\tau/\rho}$$

- Dimensional analysis

$$\frac{\mathrm{d}\bar{U}_1}{\mathrm{d}x_2} = \frac{1}{\kappa}\frac{u_\tau}{x_2}$$

$$\Rightarrow \quad \boxed{\bar{U}_1^+ = \frac{1}{\kappa}\ln\left(x_2^+\right) + C}$$

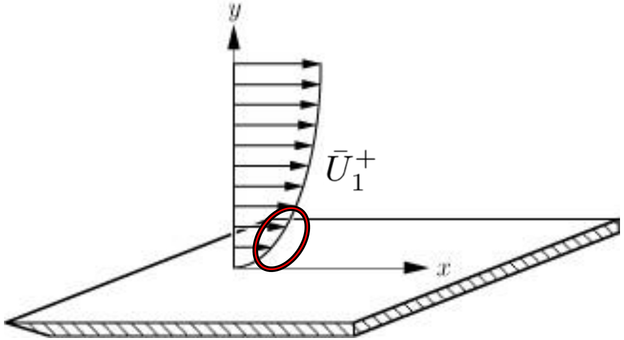


- with $x_2^+ = \frac{x_2 u_\tau}{\nu}$

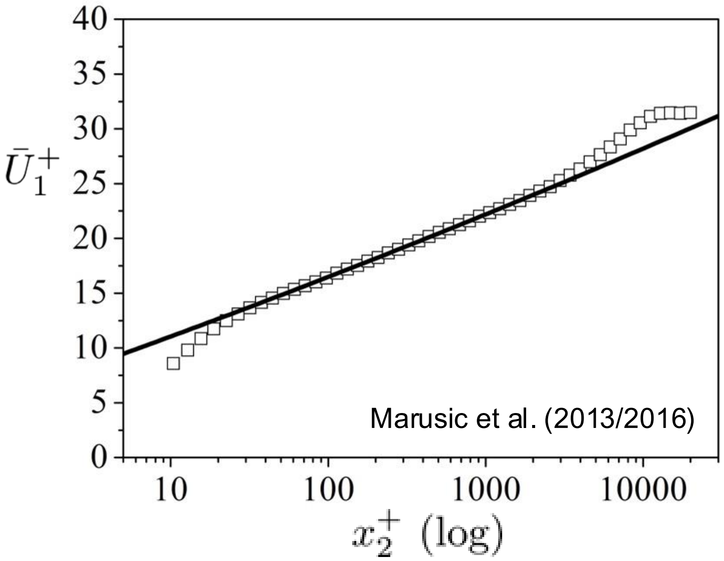
$$\bar{U}_1^+ = \frac{\bar{U}_1}{u_\tau}$$

The log-law of the wall

UNIVERSITAT POLITÈCNICA DE VALÈNCIA
TECHNISCHE UNIVERSITÄT DARMSTADT



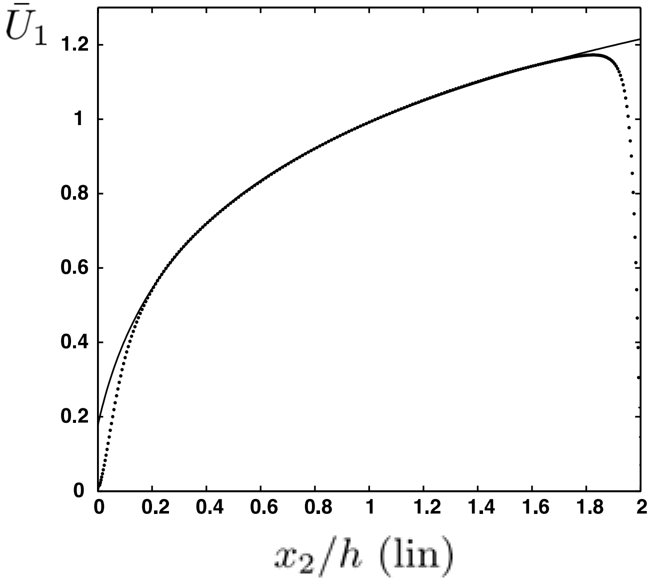
$$\boxed{\bar{U}_1^+ = \frac{1}{\kappa} \ln\left(x_2^+\right) + C}$$

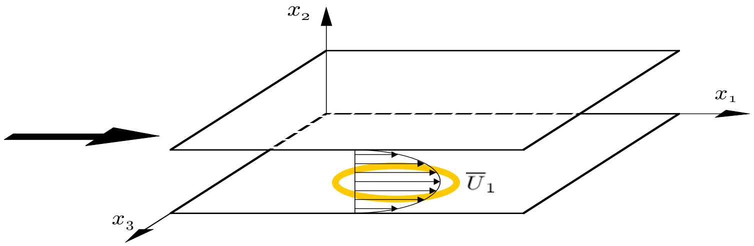


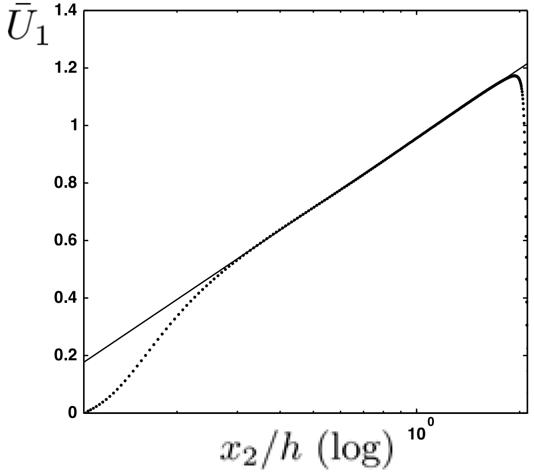
There's absolutely no connection to Navier-Stokes whatsoever!

Brainteaser: a new log-law?

- Why is there a log-law in the center?





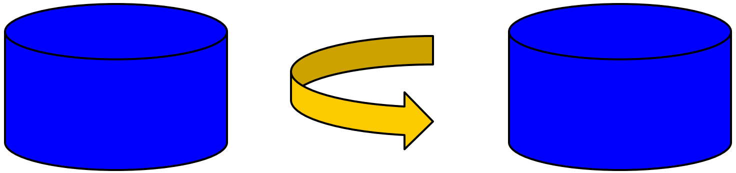


Symmetries

Concept of symmetries

Analogy between symmetric object and differential equation

- Symmetric object (under rotation)



Symmetry of differential equations

- Differential equation $\mathcal{D}(\boldsymbol{x}, \boldsymbol{u}, \boldsymbol{u}^{(1)}, \ldots) = 0$
- Symmetry transformation $\boldsymbol{x}^* = f(\boldsymbol{x}, \boldsymbol{u}) \;,\; \boldsymbol{u}^* = g(\boldsymbol{x}, \boldsymbol{u})$
- Form invariance $\mathcal{D}(\boldsymbol{x}, \boldsymbol{u}, \boldsymbol{u}^{(1)}, \ldots) = 0 \Leftrightarrow \mathcal{D}(\boldsymbol{x}^*, \boldsymbol{u}^*, \boldsymbol{u}^{*(1)}, \ldots) = 0$

Symmetry of differential equation

Example: 1D heat equation

$$\frac{\partial T}{\partial t} = \frac{\partial^2 T}{\partial x^2}$$

Symmetry transformation: scaling (two parameter)

$$S_{a_1,a_2}: \quad x^* = \mathrm{e}^{a_1} x \;, \qquad t^* = \mathrm{e}^{2a_1} t \;, \qquad T^* = \mathrm{e}^{a_2} T$$

into heat equation:

$$\Rightarrow \; \cancel{\mathrm{e}^{2a_1 - a_2}} \frac{\partial T^*}{\partial t^*} = \cancel{\mathrm{e}^{2a_1 - a_2}} \frac{\partial^2 T^*}{\partial {x^*}^2}$$

Invariants

An invariant does not change its functional form under a given symmetry transformation

Example: Scaling of 1D heat equation

$$S_{a_1,a_2}: \quad x^* = \mathrm{e}^{a_1} x \ , \quad t^* = \mathrm{e}^{2a_1} t \ , \quad T^* = \mathrm{e}^{a_2} T$$

Invariants

1) $$\tilde{x} = \frac{x}{\sqrt{t}} = \frac{\mathrm{e}^{-a_1} x^*}{\sqrt{\mathrm{e}^{-2a_1} t}} = \frac{x^*}{\sqrt{t^*}} = \tilde{x}^* \qquad \left[\frac{\mathrm{d}x}{a_1 x} = \frac{\mathrm{d}t}{2a_1 t}\right]$$

2) $$\tilde{T} = \frac{T}{t^{\frac{a_2}{2a_1}}} = \frac{\mathrm{e}^{-a_2} T^*}{\left(\mathrm{e}^{-2a_1} t^*\right)^{\frac{a_2}{2a_1}}} = \frac{T^*}{t^{*\frac{a_2}{2a_1}}} = \tilde{T}^* \qquad \left[\frac{\mathrm{d}T}{a_2 T} = \frac{\mathrm{d}t}{2a_1 t}\right]$$

Invariant solutions

- *Implementing the invariants into the DE leads to a (similarity) reduction*
- Example: 1D heat equation

$$\tilde{x} = \frac{x}{\sqrt{t}}, \qquad \tilde{T} = \frac{T}{t^{\frac{a_2}{2a_1}}}$$

- Employed as independent and dependent variables into the heat equation

$$\frac{\partial T}{\partial t} = \frac{\partial^2 T}{\partial x^2} \qquad \Rightarrow \qquad \frac{\mathrm{d}^2 \tilde{T}}{\mathrm{d}\tilde{x}^2} + \frac{1}{2}\tilde{x}\frac{\mathrm{d}\tilde{T}}{\mathrm{d}\tilde{x}} - \frac{a_2}{2a_1}\tilde{T} = 0$$

Symmetry breaking

- *Specific (boundary) conditions may adjust certain parameters*
- Boundary condition

$$x = 0 : T = T_0$$

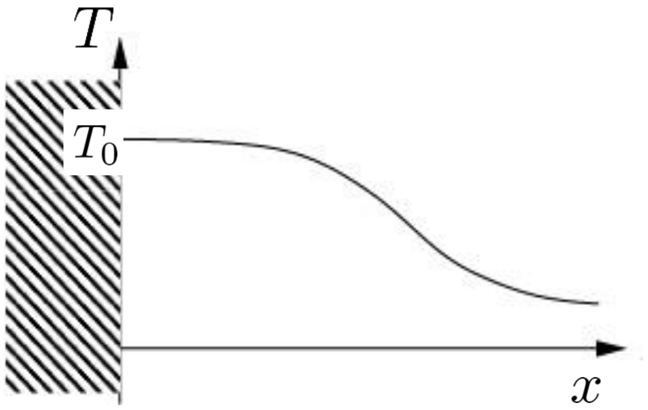
- Combined with the scaling symmetry

$$\Rightarrow \quad \mathrm{e}^{-a_1} x^* = 0 : \mathrm{e}^{-a_2} T^* = T_0$$

- Preserving the form of the BC

$$\Rightarrow \quad a_2 = 0 \, , \quad a_1$$ remains arbitrary

- T_0 is symmetry breaking



Symmetries of Navier-Stokes equations

UNIVERSITAT POLITÈCNICA DE VALÈNCIA
TECHNISCHE UNIVERSITÄT DARMSTADT

Notation

$$\frac{\partial \boldsymbol{U}}{\partial t} + (\boldsymbol{U} \cdot \nabla)\,\boldsymbol{U} = -\nabla P + \nu \Delta \boldsymbol{U}$$

$$\nabla \cdot \boldsymbol{U} = 0$$

Symmetries of Navier-Stokes I

- Translation in time:

$$T_t: \quad t^* = t + a_1 \,, \quad \boldsymbol{x}^* = \boldsymbol{x} \,, \quad \boldsymbol{U}^* = \boldsymbol{U} \,, \quad P^* = P$$

- Finite rotation:

$$T_{r_1} - T_{r_3}: \quad t^* = t \,, \quad \boldsymbol{x}^* = \mathbf{a} \cdot \boldsymbol{x} \,, \quad \boldsymbol{U}^* = \mathbf{a} \cdot \boldsymbol{U} \,, \quad P^* = P$$

Note:

- $\mathbf{a} \neq \boldsymbol{f}(t)$

(Flows are not invariant when sitting on a roundabout)

Symmetries of Navier-Stokes II

UNIVERSITAT POLITÈCNICA DE VALÈNCIA
TECHNISCHE UNIVERSITÄT DARMSTADT

- Scaling of space:

$$T_{s_1}: \quad t^* = t\,, \quad \boldsymbol{x}^* = \mathrm{e}^{a_2}\boldsymbol{x}\,, \quad \boldsymbol{U}^* = \mathrm{e}^{a_2}\boldsymbol{U}\,, \quad P^* = \mathrm{e}^{2a_2}P$$

- Scaling of time:

$$T_{s_2}: \quad t^* = \mathrm{e}^{a_3}t\,, \quad \boldsymbol{x}^* = \boldsymbol{x}\,, \quad \boldsymbol{U}^* = \mathrm{e}^{-a_3}\boldsymbol{U}\,, \quad P^* = \mathrm{e}^{-2a_3}P$$

in the limit $\frac{1}{Re} \to 0$

Symmetries of Navier-Stokes III

- Galilean invariance:

$$T_{u_1} - T_{u_3}: \quad t^* = t, \quad \boldsymbol{x}^* = \boldsymbol{x} + \boldsymbol{f}(t), \quad \boldsymbol{U}^* = \boldsymbol{U} + \frac{\mathrm{d}\boldsymbol{f}}{\mathrm{d}t}, \quad P^* = P - \quad \boldsymbol{x} \cdot \frac{\mathrm{d}^2\boldsymbol{f}}{\mathrm{d}t^2}$$

- Comprises two classical cases:
  - Translation in space: $\boldsymbol{f}(t) = \boldsymbol{a}$
  - Classical Galilean invariance: $\boldsymbol{f}(t) = \boldsymbol{b}\, t$

- A few more ... not relevant here ...

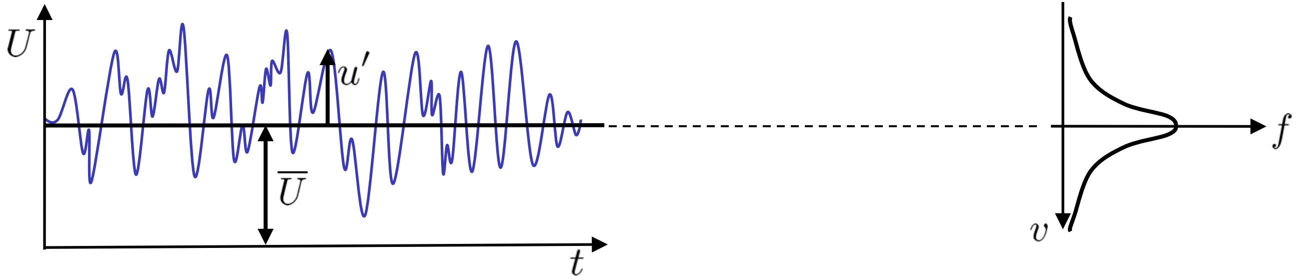
All symmetries transfer to statistical equations!

Turbulence and Symmetries

Statistics of turbulence

UNIVERSITAT POLITÈCNICA DE VALÈNCIA
TECHNISCHE UNIVERSITÄT DARMSTADT

- Randomness of turbulence



- Probability density function (PDF)

- Time average

$$\overline{\boldsymbol{U}} = \lim_{T\to\infty} \frac{1}{T} \int_0^T \boldsymbol{U}\,\mathrm{d}t$$

- 1. Moment

$$\overline{\boldsymbol{U}} = \int \boldsymbol{v} f(\boldsymbol{v},\boldsymbol{x})\,\mathrm{d}\boldsymbol{v}$$

Notation of moments

UNIVERSITAT POLITÈCNICA DE VALÈNCIA TECHNISCHE UNIVERSITÄT DARMSTADT

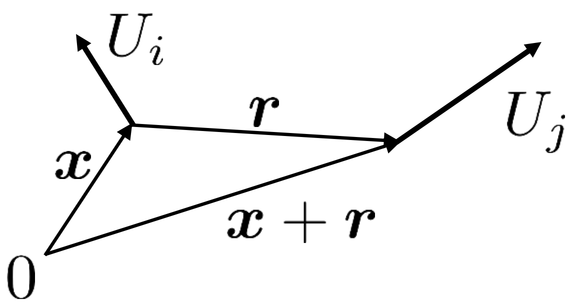
- Instantaneous value $\boldsymbol{U}, \quad P$

$$\| \qquad \|$$

- Mean value $\bar{\boldsymbol{U}}, \quad \bar{P}$

$$+ \qquad +$$

- Fluctuating quantities $\boldsymbol{u}, \quad p$



- Two-point correlation tensor (classical notation)

$$R_{ij}\left(\boldsymbol{x},\boldsymbol{r},t\right)=\overline{u_i\left(\boldsymbol{x},t\right)u_j\left(\boldsymbol{x}+\boldsymbol{r},t\right)}$$

- New definition based on instantaneous velocities

$$H_{ij}\left(\boldsymbol{x},\boldsymbol{r},t\right)=\overline{U_i\left(\boldsymbol{x},t\right)U_j\left(\boldsymbol{x}+\boldsymbol{r},t\right)}$$

Statistical moments in turbulence

- 1-point moment (mean velocity)

$$H_i(\boldsymbol{x}) = \overline{U}_i(\boldsymbol{x}) = \int \boldsymbol{v} f(\boldsymbol{v}, \boldsymbol{x}) \mathrm{d}\boldsymbol{v}$$

- 2-point moment

$$H_{ij}\left(\boldsymbol{x}_{(1)}, \boldsymbol{x}_{(2)}\right) = \overline{U_i\left(\boldsymbol{x}_{(1)}\right) U_j\left(\boldsymbol{x}_{(2)}\right)} = \int \boldsymbol{v}_{(1)} \boldsymbol{v}_{(1)} f_2 \mathrm{d}\boldsymbol{v}_{(1)} \mathrm{d}\boldsymbol{v}_{(1)}$$

- n-point moment

$$H_{i_{\{n+1\}}} = H_{i_{(0)} i_{(1)} \ldots i_{(n)}} = \overline{U_{i_{(0)}}\left(\boldsymbol{x}_{(0)}\right) \cdot \ldots \cdot U_{i_{(n)}}\left(\boldsymbol{x}_{(n)}\right)} = \ldots$$

Relations

- Non-linear relations among tensor $R_{i_{\{n+1\}}}$, $\bar{U}_i$ and $H_{i_{\{n+1\}}}$

$$
\begin{aligned}
H_{i_{(0)}} &= \bar{U}_{i_{(0)}} \\
H_{i_{(0)}i_{(1)}} &= \bar{U}_{i_{(0)}}\bar{U}_{i_{(1)}} + R_{i_{(0)}i_{(1)}} \\
H_{i_{(0)}i_{(1)}i_{(2)}} &= \bar{U}_{i_{(0)}}\bar{U}_{i_{(1)}}\bar{U}_{i_{(2)}} \\
&\quad + R_{i_{(0)}i_{(1)}}\bar{U}_{i_{(2)}} + R_{i_{(0)}i_{(2)}}\bar{U}_{i_{(1)}} + R_{i_{(1)}i_{(2)}}\bar{U}_{i_{(0)}} + R_{i_{(0)}i_{(1)}i_{(2)}} \\
\vdots \quad &\quad \vdots
\end{aligned}
$$

- Unique relation between $R_{i_{\{n+1\}}}$, $\bar{U}_i$ and $H_{i_{\{n+1\}}}$ $\Rightarrow$ we use $H_{i_{\{n+1\}}}$!

Multi-point moment equation

- Multi-point moment

$$H_{i_{\{n+1\}}} = H_{i_{(0)} i_{(1)} \ldots i_{(n)}} = \overline{U_{i_{(0)}}(\boldsymbol{x}_{(0)}) \cdot \ldots \cdot U_{i_{(n)}}(\boldsymbol{x}_{(n)})}$$

- Navier-Stokes equation → Multi-point moment equations / conservation law for moments

$$\frac{\partial U_i}{\partial t} + \frac{\partial U_i U_k}{\partial x_k} = -\frac{\partial P}{\partial x_i} + \nu \Delta U_i \;\Bigg|\; \cdot\, U_{i_{(0)}} U_{i_{(1)}} \ldots U_{i_{(n)}}$$

$$\Rightarrow \boxed{\frac{\partial H_{i_{\{n+1\}}}}{\partial t} + \sum_{l=0}^{n} \left[\frac{\partial H_{i_{\{n+2\}}[i_{(n+1)} \mapsto k_{(l)}]} \left[\boldsymbol{x}_{(n+1)} \mapsto \boldsymbol{x}_{(l)}\right]}{\partial x_{k_{(l)}}} + \frac{\partial I_{i_{\{n\}}[l]}}{\partial x_{i_{(l)}}} - \nu \frac{\partial^2 H_{i_{\{n+1\}}}}{\partial x_{k_{(l)}} \partial x_{k_{(l)}}} \right] = 0}$$

Multi-point correlation: Fluctuation approach (classical)

UNIVERSITAT POLITÈCNICA DE VALÈNCIA
TECHNISCHE UNIVERSITÄT DARMSTADT

- Definition multi-point tensor

$$R_{i_{\{n+1\}}} = R_{i_{(0)}i_{(1)}\dots i_{(n)}} = \overline{u_{i_{(0)}}(\boldsymbol{x}_{(0)}) \cdot \ldots \cdot u_{i_{(n)}}(\boldsymbol{x}_{(n)})}$$

$$\Rightarrow \frac{\partial R_{i_{\{n+1\}}}}{\partial t} + \sum_{l=0}^{n} \Bigg[\underline{\bar{U}_{k_{(l)}}(\boldsymbol{x}_{(l)}) \frac{\partial R_{i_{\{n+1\}}}}{\partial x_{k_{(l)}}}} + \underline{R_{i_{\{n+1\}}[i_{(l)} \mapsto k_{(l)}]} \frac{\partial \bar{U}_{i_{(l)}}(\boldsymbol{x}_{(l)})}{\partial x_{k_{(l)}}}}$$

$$+ \frac{\partial P_{i_{\{n\}}[l]}}{\partial x_{i_{(l)}}} - \nu \frac{\partial^2 R_{i_{\{n+1\}}}}{\partial x_{k_{(l)}} \partial x_{k_{(l)}}} - \underline{R_{i_{\{n\}}[i_{(l)} \mapsto \emptyset]} \frac{\partial \overline{u_{i_{(l)}} u_{k_{(l)}}}(\boldsymbol{x}_{(l)})}{\partial x_{k_{(l)}}}}$$

$$+ \frac{\partial R_{i_{\{n+2\}}[i_{(n+1)} \mapsto k_{(l)}]}[\boldsymbol{x}_{(n+1)} \mapsto \boldsymbol{x}_{(l)}]}{\partial x_{k_{(l)}}} \Bigg] = 0$$

- Infinite set of <u>non-linear</u> PDEs; <u>coupling</u> among all orders

New statistical symmetry I

UNIVERSITAT POLITÈCNICA DE VALÈNCIA TECHNISCHE UNIVERSITÄT DARMSTADT

$$\frac{\partial H_{i_{\{n+1\}}}}{\partial t} + \sum_{l=0}^{n} \left[\frac{\partial H_{i_{\{n+2\}}[i_{(n+1)} \mapsto k_{(l)}]}[\boldsymbol{x}_{(n+1)} \mapsto \boldsymbol{x}_{(l)}]}{\partial x_{k_{(l)}}} + \frac{\partial I_{i_{\{n\}}[l]}}{\partial x_{i_{(l)}}} - \nu \frac{\partial^2 H_{i_{\{n+1\}}}}{\partial x_{k_{(l)}} \partial x_{k_{(l)}}} \right] = 0$$

- Translation in function space:

$$\boxed{\bar{T}'_{2_{\{n\}}} : \quad t^* = t, \quad \boldsymbol{x}^* = \boldsymbol{x}, \quad \boldsymbol{r}^*_{(l)} = \boldsymbol{r}_{(l)}, \quad \mathbf{H}^*_{\{n\}} = \mathbf{H}_{\{n\}} + \mathbf{C}_{\{n\}}, \quad \mathbf{I}^*_{\{n\}} = \mathbf{I}_{\{n\}} + \mathbf{D}_{\{n\}},}$$

with $\mathbf{C}_{\{n\}}$ and $\mathbf{D}_{\{n\}}$ arbitrary constant tensors

- Key ingredient for log-law, etc. and turbulence models
- **Defines a measure of non-gaussianity**

New statistical symmetry II

$$\frac{\partial H_{i_{\{n+1\}}}}{\partial t}+\sum_{l=0}^{n}\left[\frac{\partial H_{i_{\{n+2\}}[i_{(n+1)}\mapsto k_{(l)}]}[\boldsymbol{x}_{(n+1)}\mapsto \boldsymbol{x}_{(l)}]}{\partial x_{k_{(l)}}}+\frac{\partial I_{i_{\{n\}}[l]}}{\partial x_{i_{(l)}}}-\nu\frac{\partial^2 H_{i_{\{n+1\}}}}{\partial x_{k_{(l)}}\partial x_{k_{(l)}}}\right]=0$$

- Scaling of moments:

$$\boxed{\bar{T}'_s:\quad t^*=t,\ \ \boldsymbol{x}^*=\boldsymbol{x},\ \ \boldsymbol{r}^*_{(l)}=\boldsymbol{r}_{(l)},\quad \mathbf{H}^*_{\{n\}}=\mathrm{e}^{a_s}\mathbf{H}_{\{n\}},\ \ \mathbf{I}^*_{\{n\}}=\mathrm{e}^{a_s}\mathbf{I}_{\{n\}}}$$

- **Defines a measure of intermittency**
- Important for deriving higher order moments
- Very difficult to implement into turbulence models

  $\Rightarrow$ *Klingenberg, Oberlack, Plümacher: Physics of Fluids, Phys. Fluids* ***32****, 025108 (2020)*

New statistical symmetry – PDF version

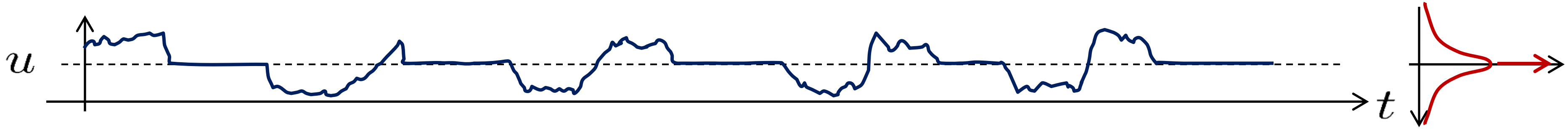
- Scaling of correlations/moments

$$\bar{T}'_s: \quad t^* = t, \quad \boldsymbol{x}^* = \boldsymbol{x}, \quad \boldsymbol{r}^*_{(l)} = \boldsymbol{r}_{(l)}, \qquad \mathbf{H}^*_{\{n\}} = \mathrm{e}^{a_s}\mathbf{H}_{\{n\}}, \quad \mathbf{I}^*_{\{n\}} = \mathrm{e}^{a_s}\mathbf{I}_{\{n\}}$$

- Equivalent symmetry of PDF (LMN-eqn)

$$f_n^* = \delta(\boldsymbol{v}_{(1)}, \ldots, \boldsymbol{v}_{(n)}) + \mathrm{e}^{a_s}\left(f_n - \delta(\boldsymbol{v}_{(1)}, \ldots, \boldsymbol{v}_{(n)})\right)$$

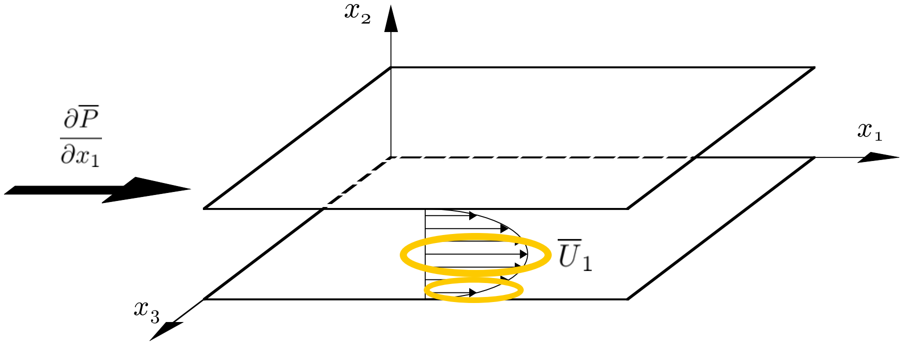
- Consequences
  - Mathematically: since f_n and f_n^* are non-negative functions $\Rightarrow a_s \geq 0$ (semi-group)
  - Physically: measure of intermittency



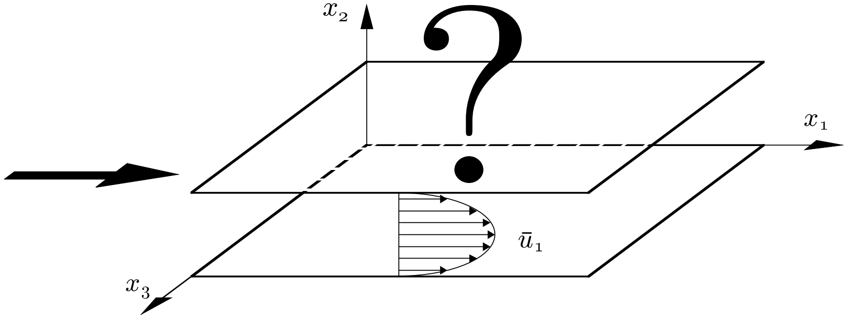
Symmetry induced scaling laws

Regions of turbulent scaling laws

- Channel flow scaling laws

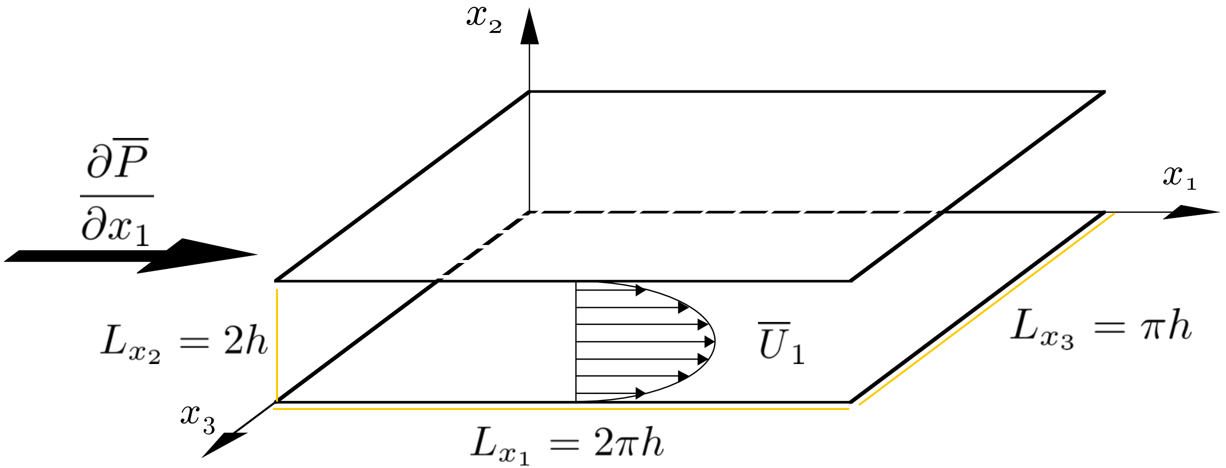


- Brain teaser - “modified“ channel flow



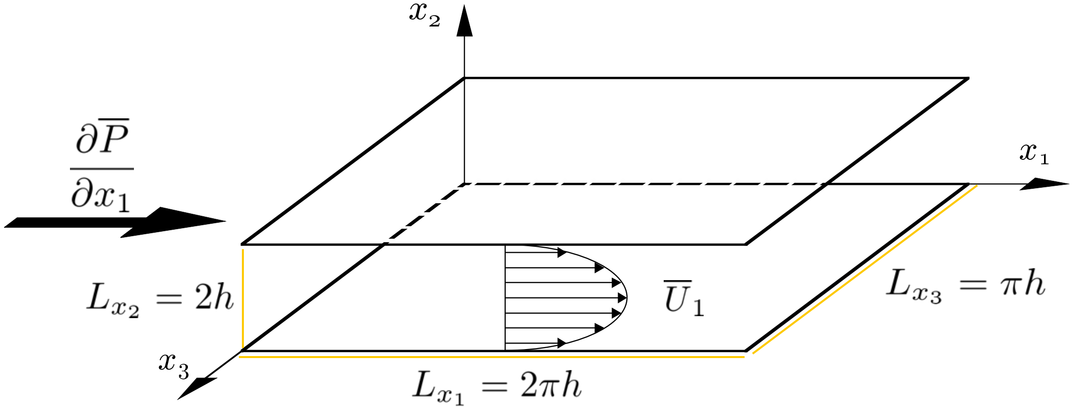
Channel flow DNS details I

- Pure Poiseuille channel flow @ $\mathrm{Re}_\tau = 10^4$
- Conducted at SuperMuc, LRZ, Munich, Germany
- Code: LISO (Hoyas & Jimenez, 2006)
- Fourier in x_1 and x_3, compact finite differences in x_2, Runge-Kutta time-stepper

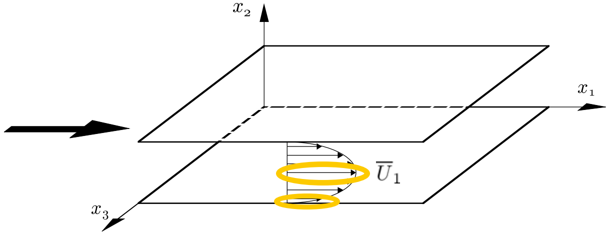


Channel flow DNS details II

Re_τ	Δx_1^+	Δx_3^+	$\Delta_{max} x_2^+$	L_{x_1}/h	L_{x_3}/h	$\Delta t u_\tau/h$	Ref.	
2000	12.3	6.2	8.9	8π	3π	10.3	HJ06	
4200	12.8	6.4	10.7	2π	π	15.0	LJ14	
5200	12.7	6.4	10.3	8π	3π	7.8	LM15	
10000	15.3	7.6	13.0	2π	π	14.10	HO22 (PRF)	~ 60M CPU h



Specific symmetries for shear flows



- Fundamental geometrical assumption for the flow

$$\boxed{\bar{\boldsymbol{U}} = [\bar{U}_1(x_2), 0, 0] \ , \ \ldots \ , \ \overline{U^n_{[i]}} = \overline{U^n_{[i]}}(x_2)}$$

- Related symmetries:
  - Scaling:

$$T_{s_1-s_3}: \quad (t^* = e^{a_{St}}\, t),\ x_2^* = e^{a_{Sx}}\, x_2,\ U_1^* = e^{a_{Sx}-a_{St}+a_{Ss}}\, U_1,$$
$$U^2_{[i]}{}^* = e^{2(a_{Sx}-a_{St})+a_{Ss}} U^2_{[i]} \ , \ \ldots \ , \ \overline{U^n_{[i]}}{}^* = e^{n(a_{Sx}-a_{St})+a_{Ss}} \overline{U^n_{[i]}}$$

  - Translation in function space:

$$T_{\bar{U}_1}: \quad x_2^* = x_2 \ , \ \bar{U}_1^* = \bar{U}_1 + a_1 \ , \ \overline{U^n_{[i]}}{}^* = \overline{U^n_{[i]}} + a_{i_{\{n\}}} \ , \ \ldots$$

  - Translational invariance in x_2 – direction:

$$T_{x_2}: \quad x_2^* = x_2 + a_4 \ , \ \bar{U}_1^* = \bar{U}_1 \ , \ \overline{U^n_{[i]}}{}^* = \overline{U^n_{[i]}} \ , \ \ldots$$

Wall-bounded parallel shear flows

UNIVERSITAT POLITÈCNICA DE VALÈNCIA
TECHNISCHE UNIVERSITÄT DARMSTADT

- Invariance condition / scaling law equations

$$\boxed{\frac{\mathrm{d}\bar{U}_1}{\mathrm{d}x_2} = \frac{[(a_{Sx} - a_{St}) + a_{Ss}]\bar{U}_1 + a^H_{1_{\{1\}}}}{a_{Sx}x_2 + a_{x_2}}, \quad \dots \quad, \frac{\mathrm{d}\overline{U^n_{[i]}}}{\mathrm{d}x_2} = \frac{[n(a_{Sx} - a_{St}) + a_{Ss}]\overline{U^n_{[i]}} + a^H_{i_{\{n\}}}}{a_{Sx}x_2 + a_{x_2}}}$$

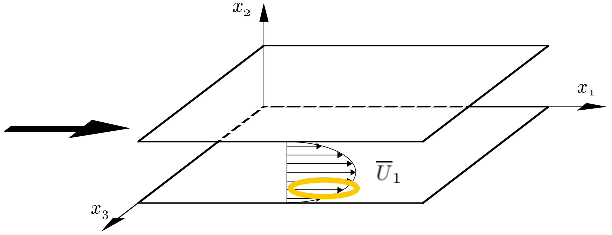
- $a_{Ss}, a^H_{1_{\{1\}}}, \dots, a^H_{1_{\{n\}}}$ are the group parameter of the statistical symmetries
- Recall *Prandtl / von Karman* equation

$$\frac{\mathrm{d}\bar{U}_1}{\mathrm{d}x_2} = \frac{1}{\kappa}\frac{u_\tau}{x_2}$$

Log-region – 1st & higher moments

UNIVERSITAT POLITÈCNICA DE VALÈNCIA
TECHNISCHE UNIVERSITÄT DARMSTADT

- Symmetry breaking: wall-friction velocity $u_\tau = \sqrt{\frac{\tau_w}{\rho}}$



$$x_2^* = \mathrm{e}^{a_{Sx}}\, x_2 \ , \quad t^* = \mathrm{e}^{a_{St}}\, t \ , \quad \bar{U}_1^* = \mathrm{e}^{\overbrace{a_{Sx} - a_{St} + a_{Ss}}^{=0}}\, \bar{U}_1 \ , \ \dots$$

- Solution

$$\frac{\mathrm{d}\bar{U}_1}{\mathrm{d}x_2} = \frac{\overbrace{[(a_{Sx} - a_{St}) + a_{Ss}]}^{=0}\bar{U}_1 + a^H_{1_{\{1\}}}}{a_{Sx}x_2 + a_{x_2}} , \ \dots \ , \ \frac{\mathrm{d}\overline{U^n_{[i]}}}{\mathrm{d}x_2} = \frac{[n(a_{Sx} - a_{St}) + a_{Ss}]\overline{U^n_{[i]}} + a^H_{i_{\{n\}}}}{a_{Sx}x_2 + a_{x_2}}$$

$$\boxed{\bar{U}_1^+ = \frac{1}{\kappa}\ln\left(x_2^+ + A^+\right) + B} \qquad \boxed{\overline{U^n_{[i]}}^+ = C_{[i]_{\{n\}}}\left(x_2^+ + A^+\right)^{\omega(n-1)} - B_{[i]_{\{n\}}}} \quad n \geq 2$$

Symmetry reduction

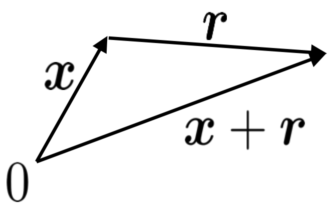
Symmetry reduction log-region I

- Multi-point moment equation

$$\frac{\partial H_{i_{\{n\}}}}{\partial t}+\sum_{l=0}^{n}\left[\frac{\partial H_{i_{\{n+1\}}[i_{(n)}\mapsto k_{(l)}]}\left[\boldsymbol{x}_{(n)}\mapsto\boldsymbol{x}_{(l)}\right]}{\partial x_{k_{(l)}}}+\frac{\partial I_{i_{\{n-1\}}[l]}}{\partial x_{i_{(l)}}}-\nu\frac{\partial^2 H_{i_{\{n\}}}}{\partial x_{k_{(l)}}\partial x_{k_{(l)}}}\right]=0$$

- Invariants / similarity variables

$$\tilde{\boldsymbol{r}}_{(l)}=\frac{\boldsymbol{r}_{(l)}}{x_2+A}$$



$$H_{i_{\{n\}}}=\tilde{H}_{i_{\{n\}}}\underbrace{\left(\tilde{\boldsymbol{r}}_{(2)},\ldots,\tilde{\boldsymbol{r}}_{(n)}\right)}\cdot\left(x_2+\frac{a_{x_2}}{a_{Sx}}\right)^{\omega(n-1)}-\tilde{B}_{i_{\{n\}}}$$

$$\tilde{\boldsymbol{r}}_{(l)}=0\quad\Rightarrow\quad\overline{U^n_{[i]}}^{+}=C_{[i]_{\{n\}}}\left(x_2^{+}+A^{+}\right)^{\omega(n-1)}-B_{[i]_{\{n\}}}$$

Symmetry reduction log-region II

- Symmetry reduced infinite multi-point moment equation

$$0 = \left(n\omega - \sum_{l=2}^{n} \tilde{r}_{m(l)} \frac{\partial}{\partial \tilde{r}_{m(l)}}\right) \tilde{H}_{i_{\{n+1\}}[i_{(n+1)}\mapsto 2]}\left(\tilde{\boldsymbol{r}}_{(2)}, \ldots, \tilde{\boldsymbol{r}}_{(n)}, 0\right) + \delta_{i(1)2}\left(n\omega - \sum_{l=2}^{n} \tilde{r}_{m(l)} \frac{\partial}{\partial \tilde{r}_{m(l)}}\right) I_{i\{n-1\}[1]}\left(\tilde{\boldsymbol{r}}_{(2)}, \ldots, \tilde{\boldsymbol{r}}_{(n)}\right)$$

$$+\sum_{l=2}^{n}\Bigg[\left(\frac{\partial \tilde{H}_{i_{\{n+1\}}[i_{(n+1)}\mapsto k_{(l)}]}\left(\tilde{\boldsymbol{r}}_{(2)}, \ldots, \tilde{\boldsymbol{r}}_{(n+1)}\right)}{\partial \tilde{r}_{k(l)}} + \frac{\partial \tilde{H}_{i_{\{n+1\}}[i_{(n+1)}\mapsto k_{(l)}]}\left(\tilde{\boldsymbol{r}}_{(2)}, \ldots, \tilde{\boldsymbol{r}}_{(n+1)}\right)}{\partial \tilde{r}_{k(n+1)}}\right)\Bigg|_{\tilde{\boldsymbol{r}}_{(n+1)}\mapsto \tilde{\boldsymbol{r}}_{l}}$$

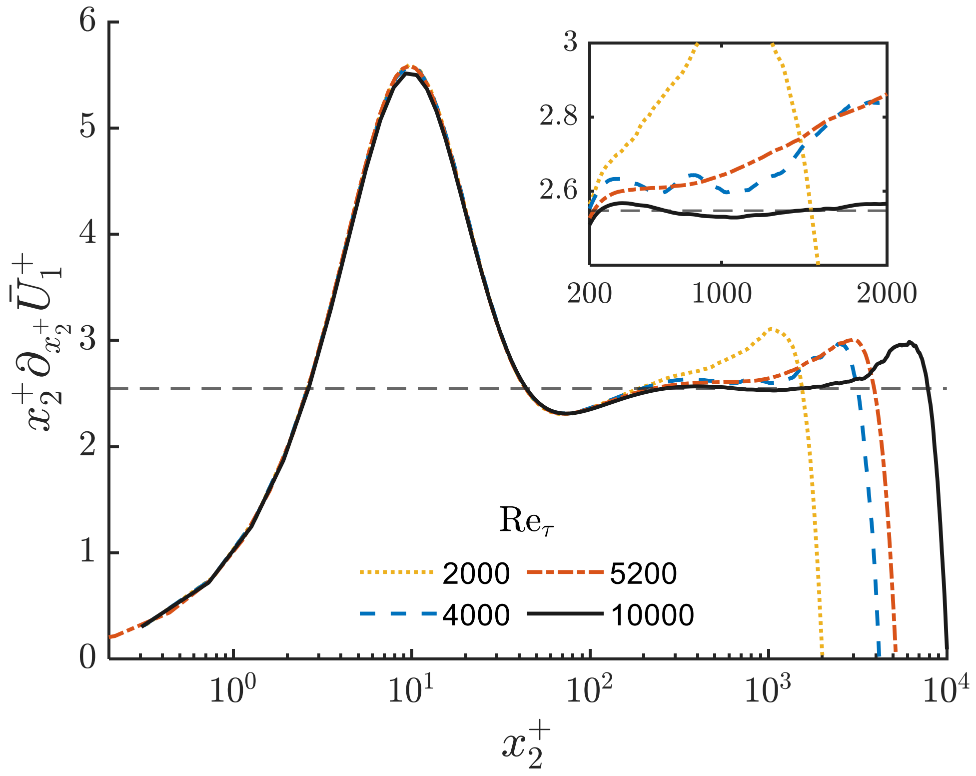
$$-\left(\frac{\partial \tilde{H}_{i_{\{n+1\}}[i_{(n+1)}\mapsto k_{(l)}]}\left(\tilde{\boldsymbol{r}}_{(2)}, \ldots, \tilde{\boldsymbol{r}}_{(n+1)}\right)}{\partial \tilde{r}_{k(l)}}\right)\Bigg|_{\tilde{\boldsymbol{r}}_{(n+1)}\mapsto 0} + \frac{\partial \tilde{I}_{i_{\{n-1\}}}[l]\left(\tilde{\boldsymbol{r}}_{(2)}, \ldots, \tilde{\boldsymbol{r}}_{(n)}\right)}{\partial \tilde{r}_{i(l)}} - \frac{\partial \tilde{I}_{i_{\{n-1\}}}[1]\left(\tilde{\boldsymbol{r}}_{(2)}, \ldots, \tilde{\boldsymbol{r}}_{(n)}\right)}{\partial \tilde{r}_{i(l)}}\Bigg].$$

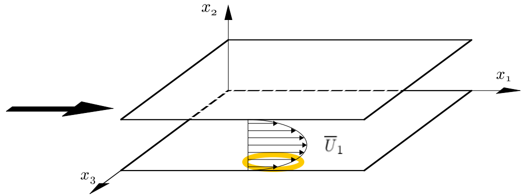
Scaling laws verification

TECHNISCHE UNIVERSITÄT DARMSTADT

UNIVERSITAT POLITÈCNICA DE VALÈNCIA

Log-region – mean velocity scaling

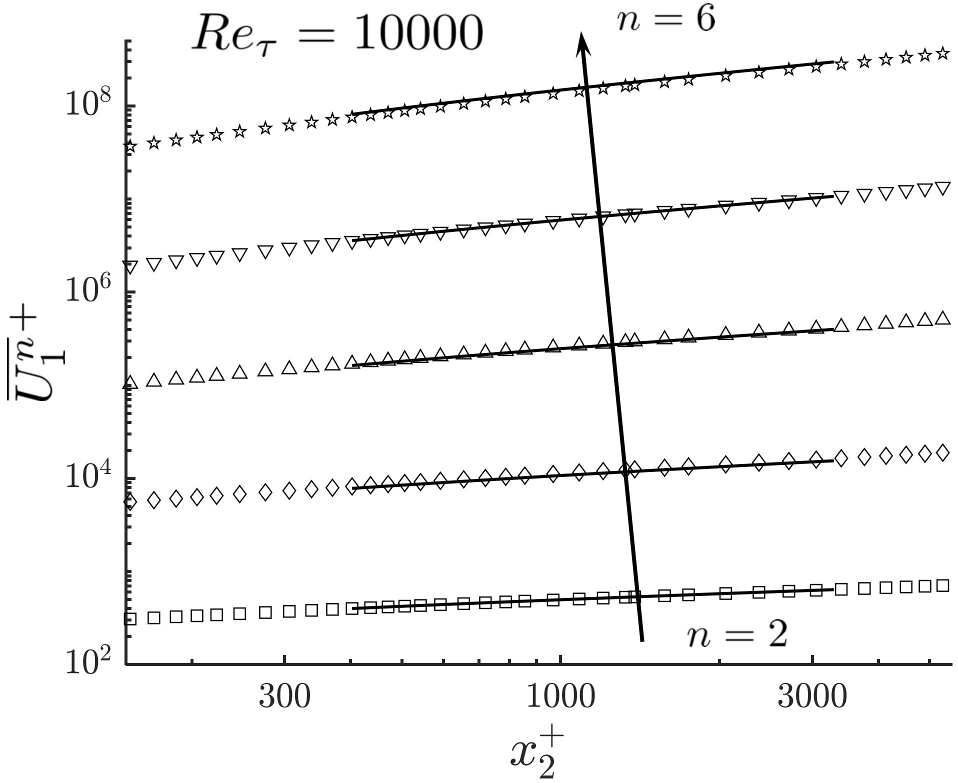


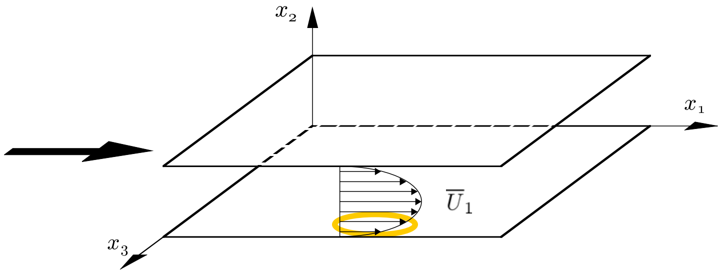


Re_τ	κ	B
1000	0.39	4.51
2000	0.37	3.85
4000	0.38	4.27
5200	0.38	4.26
10000	0.39	4.44

$$\boxed{\overline{U}_1^+ = \frac{1}{\kappa} \ln\left(x_2^+ + A\right) + B}$$

Log-region – U_1-moments scaling



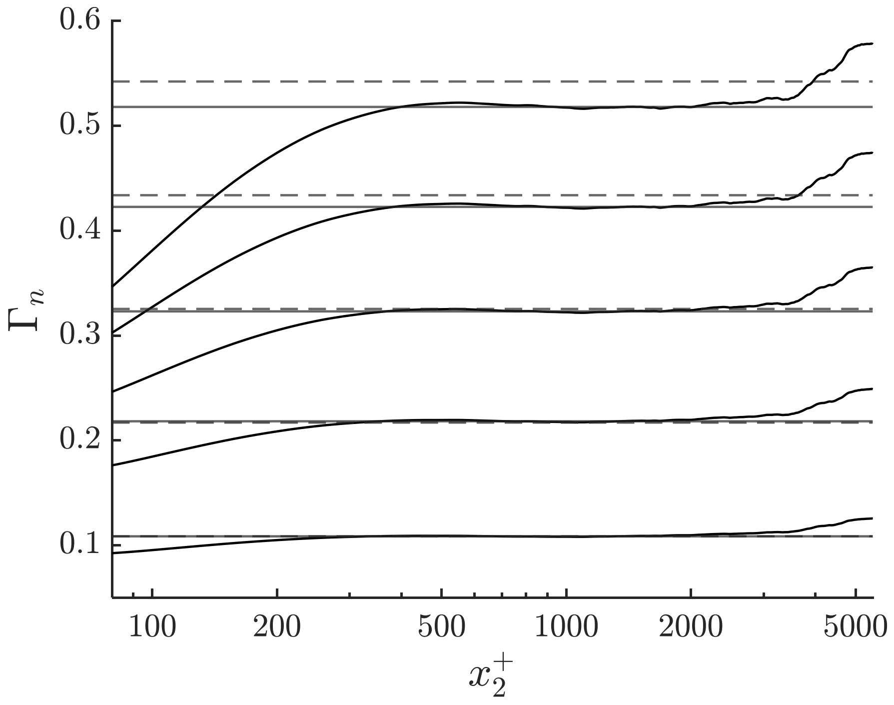


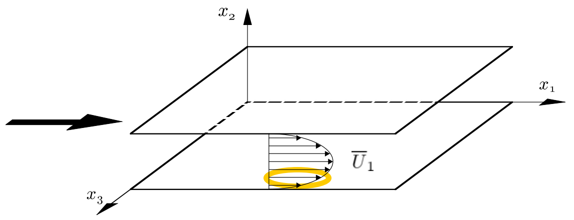
n	DNS	$\omega(n-1)$
2	0.10	-
3	0.20	0.20
4	0.30	0.30
5	0.40	0.40
6	0.49	0.50

$$\overline{U_1^n}^+ = C_{[1]_{\{n\}}} \left(x_2^+ + A^+\right)^{\omega(n-1)} - B_{[1]_{\{n\}}}$$

Log-region – moments indicator funct.

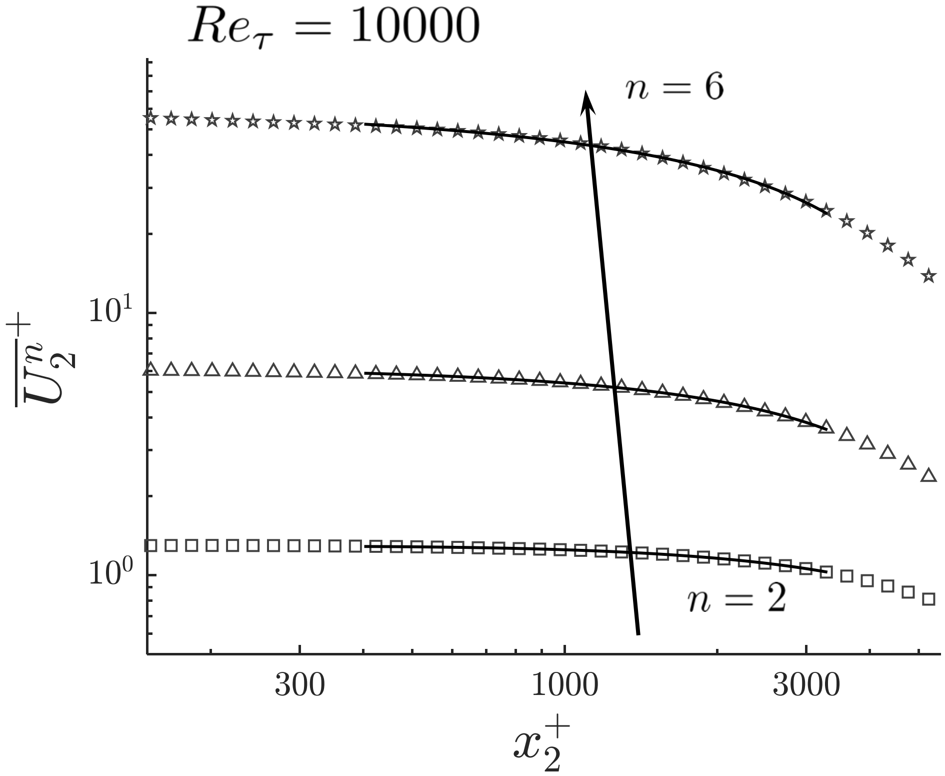
UNIVERSITAT POLITÈCNICA DE VALÈNCIA
TECHNISCHE UNIVERSITÄT DARMSTADT

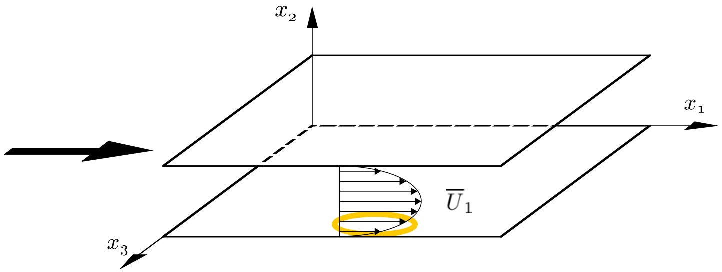




$$\Gamma_n = \frac{x_2^+}{\overline{U_1^n}^+ + B_n} \frac{\mathrm{d}\overline{U_1^n}^+}{\mathrm{d}x_2^+} = \omega(n-1)$$

Log-region – U_2-moments scaling

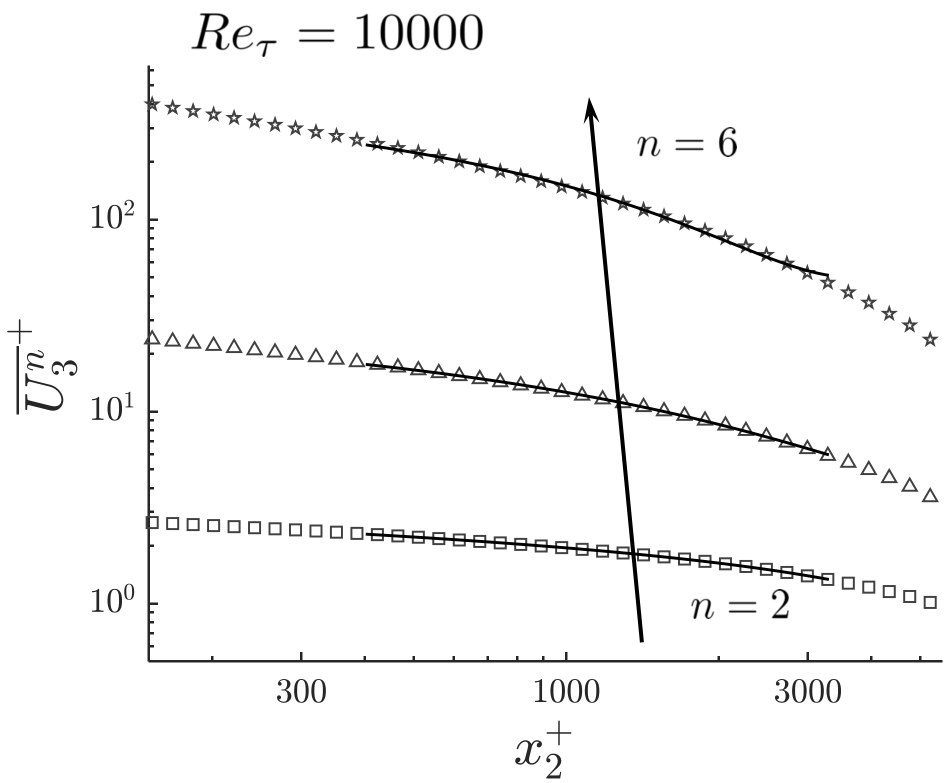


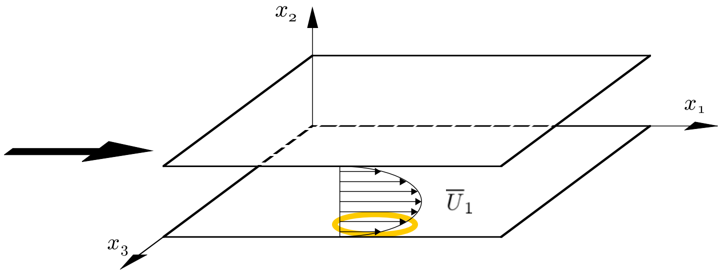


$$\omega = 0.1$$

$$\boxed{\overline{U_2^n}^+ = C_{[2]_{\{n\}}} \left(x_2^+ + A^+\right)^{\omega(n-1)} - B_{[2]_{\{n\}}}}$$

Log-region – U_3-moments scaling

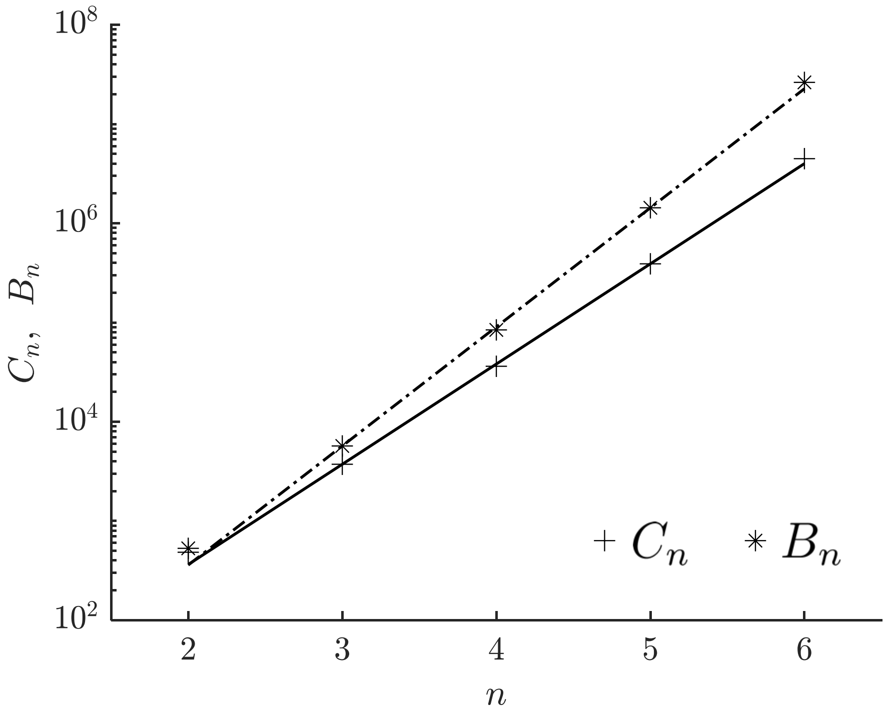


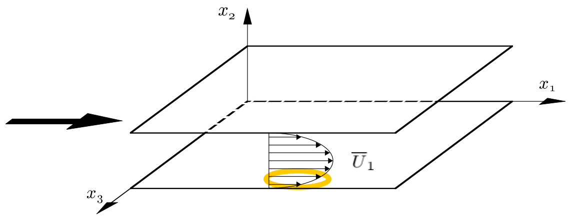


$$\omega = 0.1$$

$$\overline{U_3^n}^+ = C_{[3]_{\{n\}}} \left(x_2^+ + A^+\right)^{\omega(n-1)} - B_{[3]_{\{n\}}}$$

Log-region – prefct. scaling of C_n, B_n





$$\overline{U_1^n}^{+} = C_n \left(x_2^+ + A^+\right)^{\omega(n-1)} - B_n$$

$$C_n = \alpha e^{\beta n}, \; B_n = \tilde{\alpha} e^{\tilde{\beta} n}$$

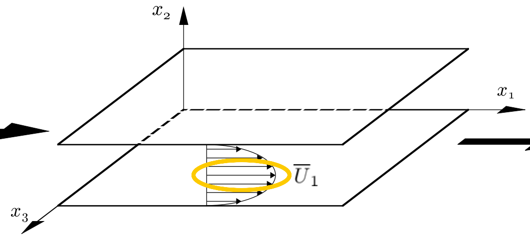
$\alpha = 3.48$ $\quad$ $\tilde{\alpha} = 1.43$

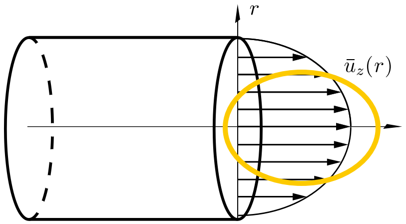
$\beta = 2.33$ $\quad$ $\tilde{\beta} = 2.76$

Centre / deficit region

- Symmetry breaking quantity unclear

$$\Rightarrow a_{Sx} - a_{St} + a_{Ss} \neq 0$$





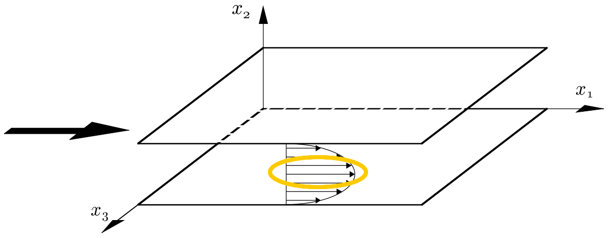
- Scale invariance

$$x_2^* = \mathrm{e}^{a_{Sx}}\, x_2 \ , \quad t^* = \mathrm{e}^{a_{St}}\, t \ , \quad \bar{U}_1^* = \mathrm{e}^{\overbrace{a_{Sx} - a_{St} + a_{Ss}}^{\neq 0}}\, \bar{U}_1 \ , \quad \ldots$$

- Invariance condition

$$\frac{\mathrm{d}\bar{U}_1}{\mathrm{d}x_2} = \frac{\overbrace{[(a_{Sx} - a_{St}) + a_{Ss}]}^{\neq 0}\,\bar{U}_1 + a^H_{1_{\{1\}}}}{a_{Sx}x_2 + a_{x_2}}, \quad \ldots \quad , \frac{\mathrm{d}\overline{U_1^n}}{\mathrm{d}x_2} = \frac{[n(a_{Sx} - a_{St}) + a_{Ss}]\overline{U_1^n} + a^H_{1_{\{n\}}}}{a_{Sx}x_2 + a_{x_2}}$$

Centre / deficit region scaling laws



- 1st moment / mean velocity

$$\boxed{\frac{\bar{U}_1^{(0)} - \bar{U}_1}{u_\tau} = C_1' \left(\frac{x_2}{h}\right)^{\sigma_1}}$$

- 2nd moment

$$\boxed{\frac{\overline{U_1^2}^{(0)} - \overline{U_1^2}}{u_\tau^2} = C_2' \left(\frac{x_2}{h}\right)^{\sigma_2}}$$

- Moments for $n > 2$

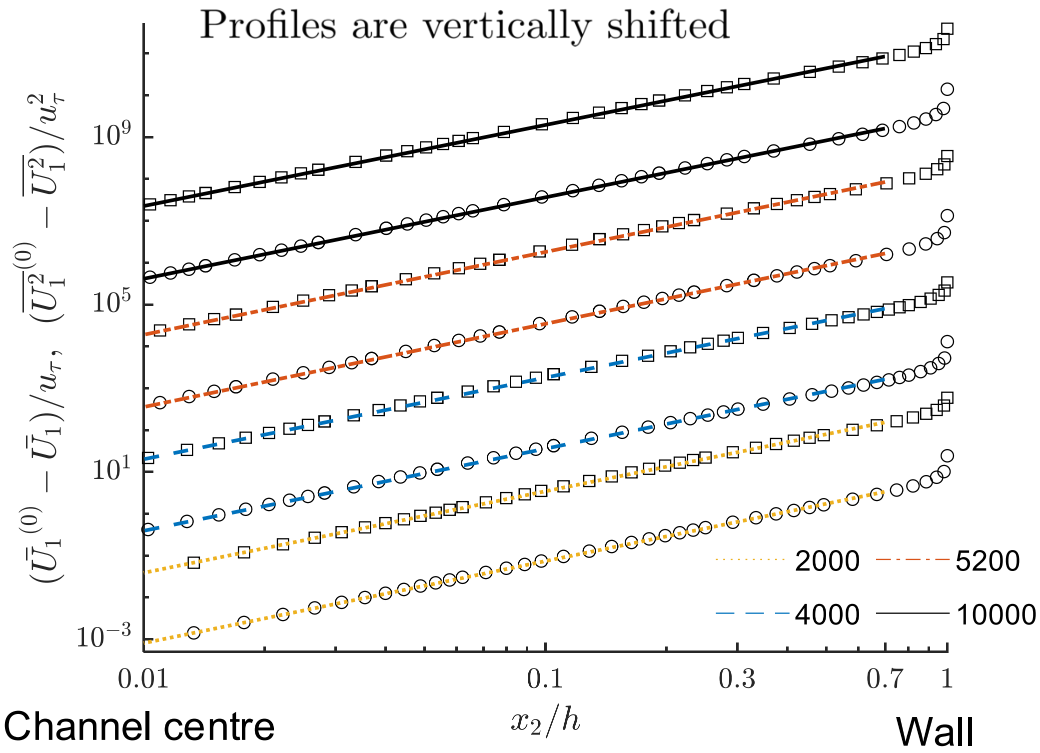
$$\boxed{\frac{\overline{U_1^n}^{(0)} - \overline{U_1^n}}{u_\tau^n} = C_n' \left(\frac{x_2}{h}\right)^{n(\sigma_2-\sigma_1)+2\sigma_1-\sigma_2}}$$

with

$$\boxed{C_n' = \alpha' \mathrm{e}^{\beta' n}}$$

DNS: Centre region 1st & 2nd moment scaling

UNIVERSITAT POLITÈCNICA DE VALÈNCIA
TECHNISCHE UNIVERSITÄT DARMSTADT

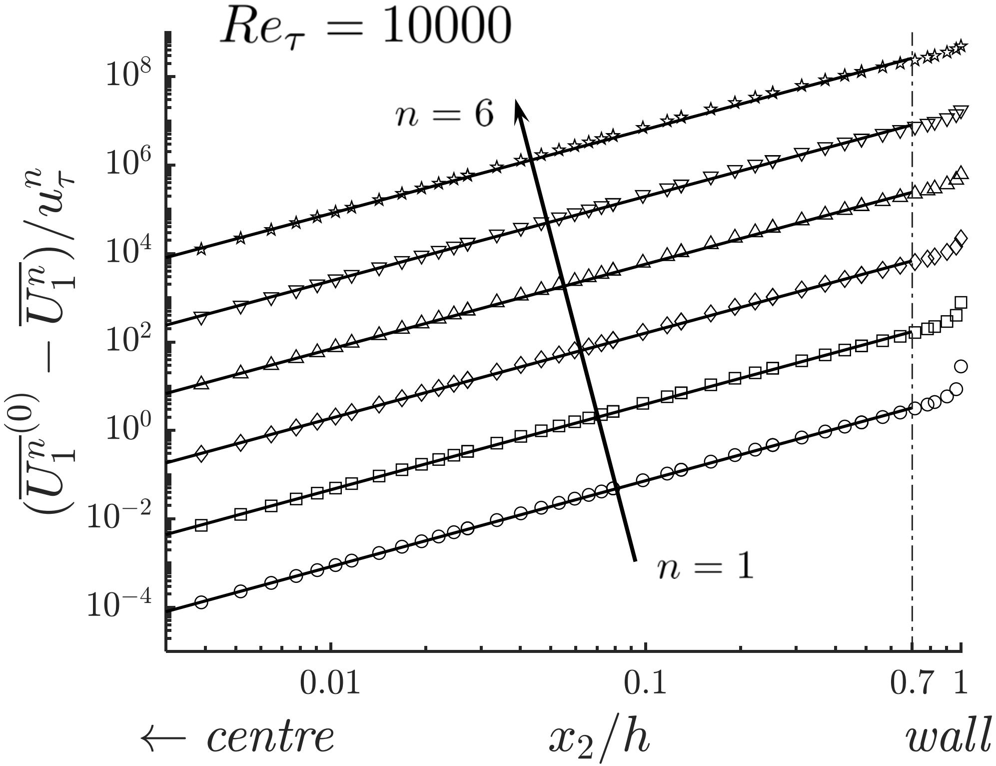


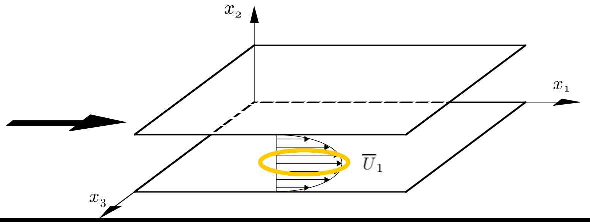
$$\frac{\bar{U}_1^{(0)} - \bar{U}_1}{u_\tau} = C_1' \left(\frac{x_2}{h}\right)^{\sigma_1}$$

$$\frac{\overline{U_1^2}^{(0)} - \overline{U_1^2}}{u_\tau^2} = C_2' \left(\frac{x_2}{h}\right)^{\sigma_2}$$

Re_τ	C_1'	C_2'	σ_1	σ_2
2000	6.75	307	1.96	1.95
4000	6.58	320	1.96	1.95
5200	6.74	343	1.99	1.98
10000	6.44	339	1.95	1.94

DNS: Centre region nth moment scaling





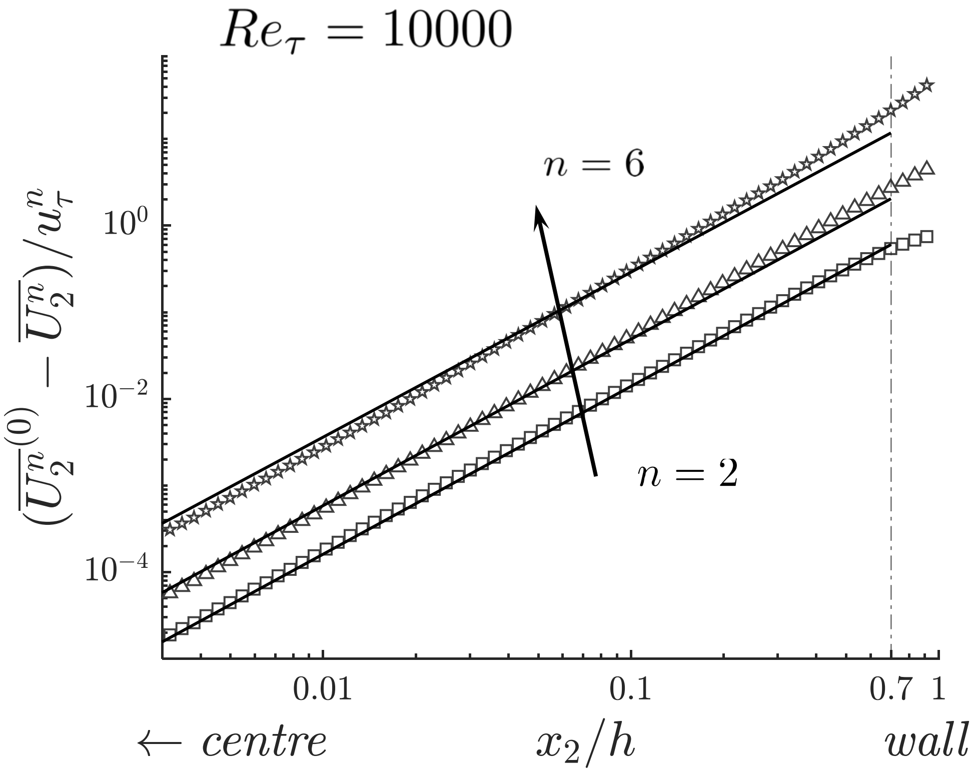
$$\boxed{\frac{\overline{U_1^n}(0) - \overline{U_1^n}}{u_\tau^n} = C'_n \left(\frac{x_2}{h}\right)^{n(\sigma_2-\sigma_1)+2\sigma_1-\sigma_2}}$$

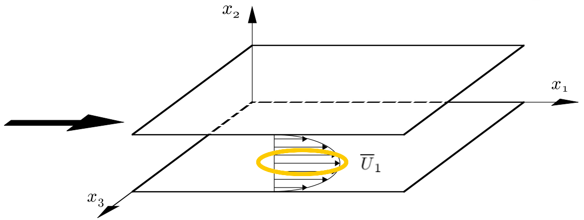
n	DNS	$n(\sigma_2 - \sigma_1)$ $+2\sigma_1 - \sigma_2$
1	1.95	-
2	1.94	-
3	1.93	1.93
4	1.92	1.92
5	1.91	1.91
6	1.91	1.90

Value almost zero

$\Rightarrow$ strongly intermittent!

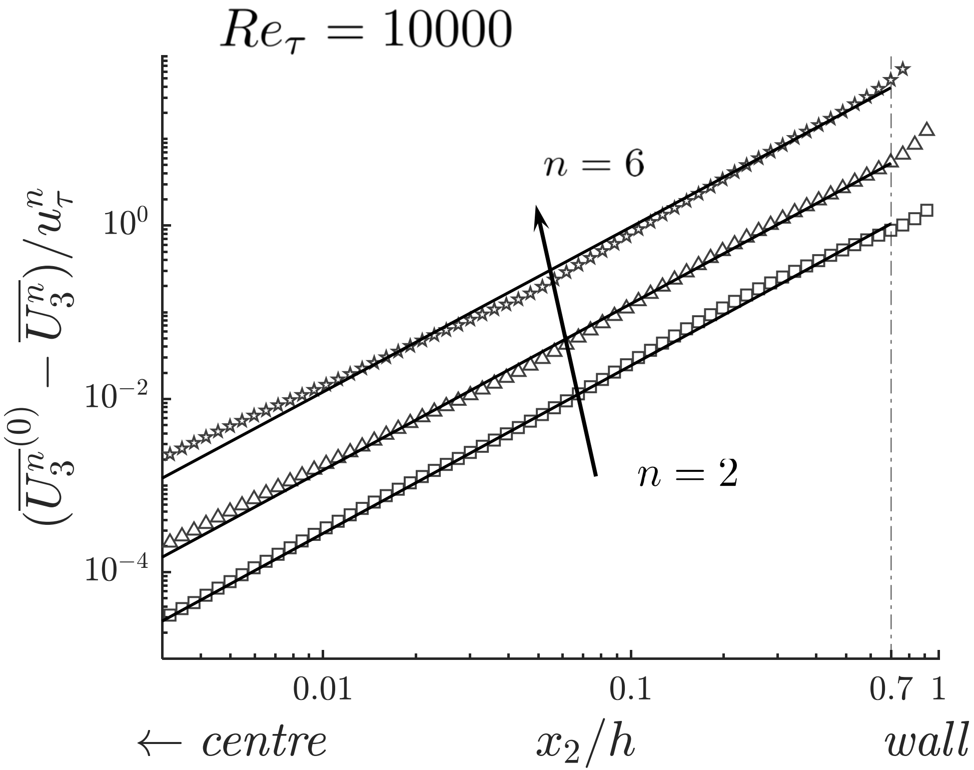
Centre region nth U_2-moment scaling

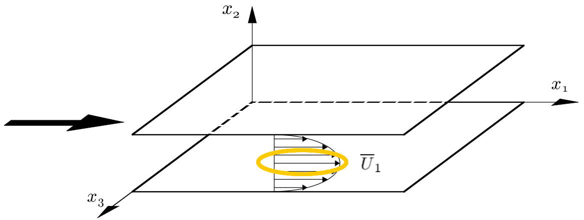




$$\frac{\overline{U_2^n}^{(0)} - \overline{U_2^n}}{u_\tau^n} = C_n' \left(\frac{x_2}{h}\right)^{n(\sigma_2-\sigma_1)+2\sigma_1-\sigma_2}$$

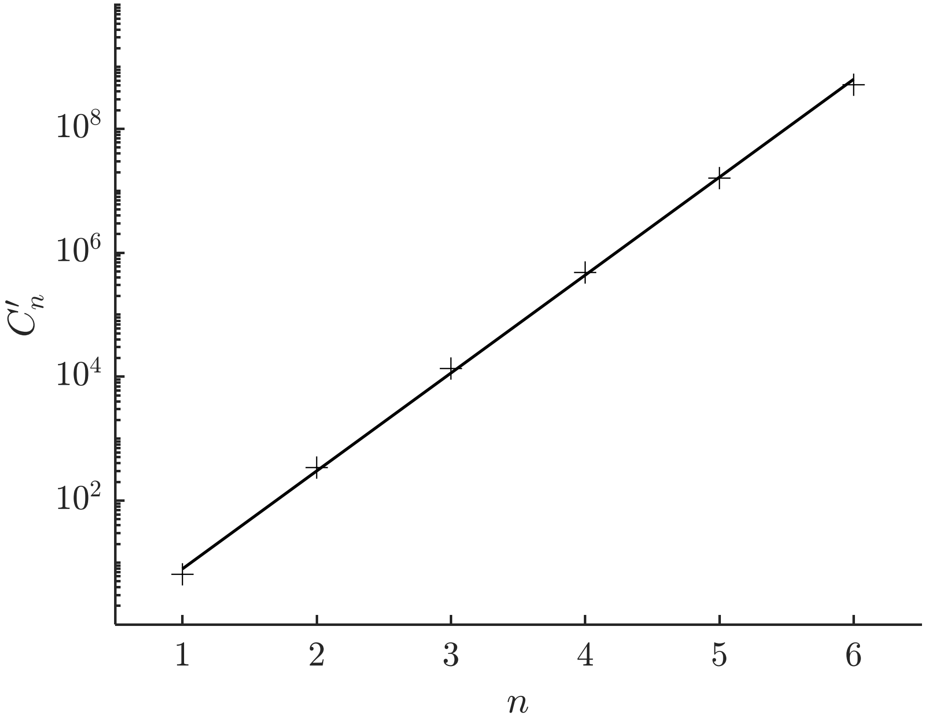
Centre region n[th] U_3-moment scaling

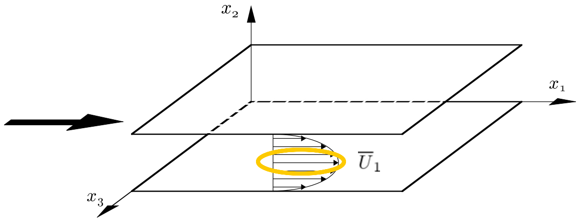




$$\boxed{\frac{\overline{U_3^n}^{(0)} - \overline{U_3^n}}{u_\tau^n} = C'_n \left(\frac{x_2}{h}\right)^{n(\sigma_2-\sigma_1)+2\sigma_1-\sigma_2}}$$

Centre region – prefactor scaling of C'_n



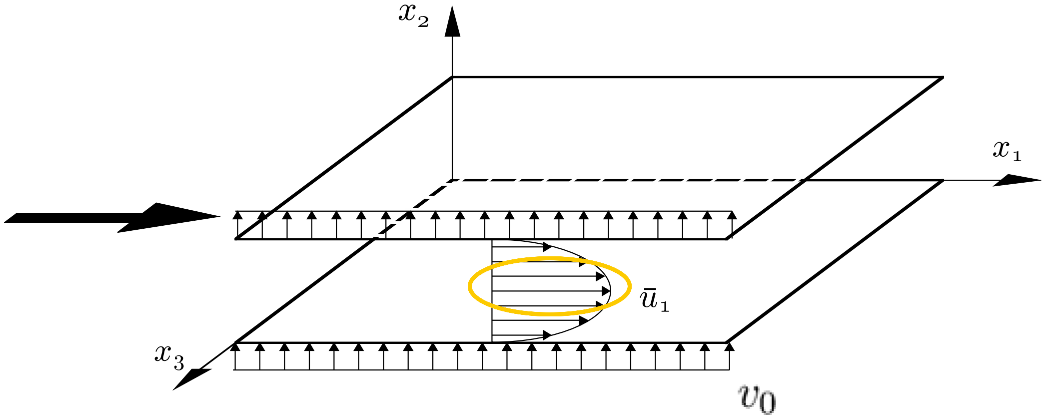


$$\boxed{\frac{\overline{U_1^n}(0) - \overline{U_1^n}}{u_\tau^n} = C'_n \left(\frac{x_2}{h}\right)^{n(\sigma_2-\sigma_1)+2\sigma_1-\sigma_2}}$$

$$\boxed{C'_n = \alpha' e^{\beta' n}}$$

$$\alpha' = 0.21 \qquad \beta' = 3.64$$

Solution to the "brain teaser": Channel flow with transpiration



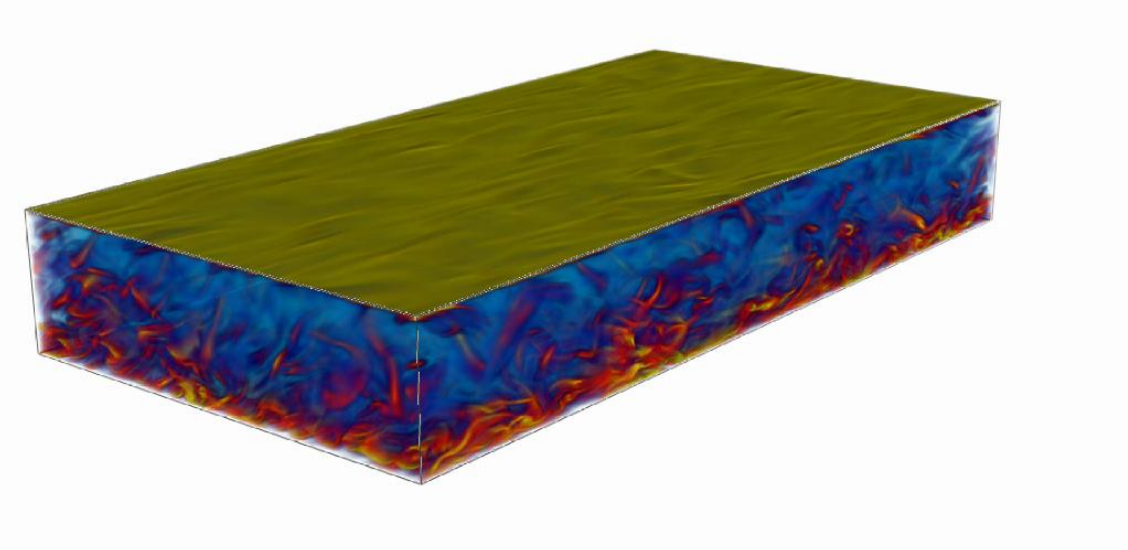
- Symmetry suggested solution for the centre region

$$\bar{U}_1(x_2) = \hat{u}_1 \ln\left[\alpha\left(\frac{x_2}{h} + \beta\right)\right] + \hat{u}_2$$

- Avsarkisov, Oberlack, Hoyas: J. Fluid Mech. (2014), 746

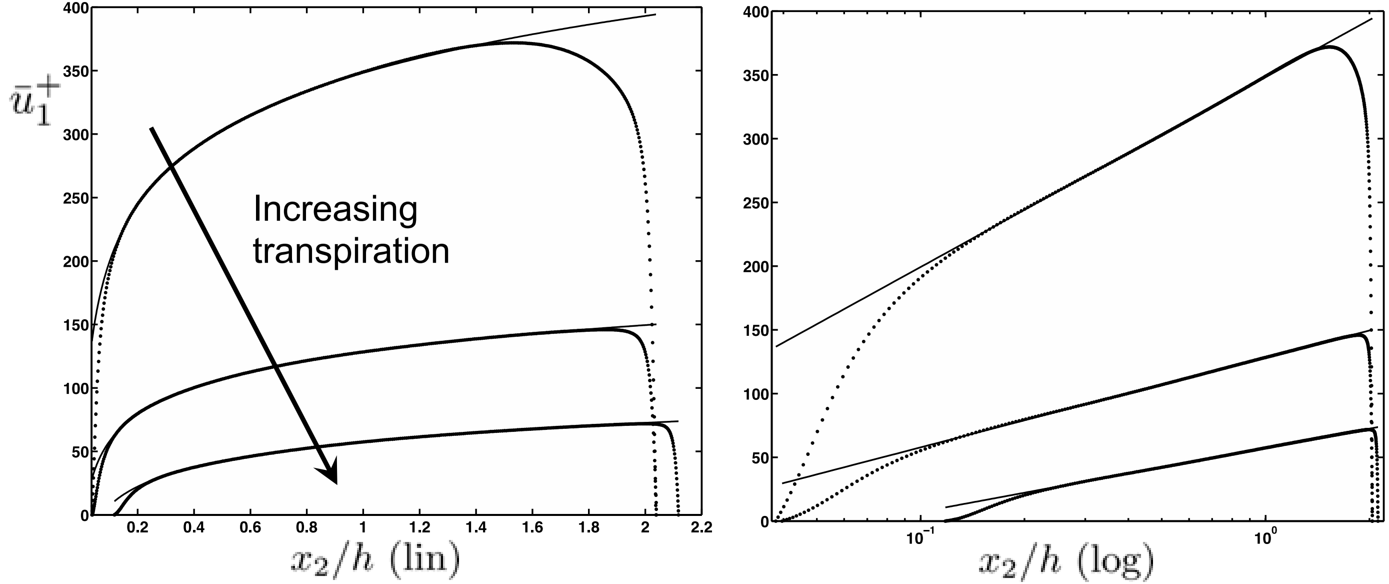
Channel flow with transpiration

- Vorticity magnitude



- Avsakisov et al. J. Fluid Mech. (2014), vol. 746, pp. 99-122.

Channel flow with transpiration



$$\frac{\bar{U}_1(x_2) - U_{\text{bulk}}}{u_\tau} = \frac{1}{\gamma} \ln \left[\left(\frac{x_2}{h} + \beta \right) \right]$$

Conclusions

UNIVERSITAT POLITÈCNICA DE VALÈNCIA
TECHNISCHE UNIVERSITÄT DARMSTADT

- Basis for the analysis is the *H*-moment equations
- Sets of statistical symmetries have been derived - hidden in Navier-Stokes equations
- Statistical symmetries mirror key properties of turbulence: *intermittency* and *non-gaussianity*
- They are the key ingredients for classical and new high-moment scaling laws
- Statistical symmetries give rise to anomalous scaling
- In particular:

  ⇒ Scaling laws are invariant solutions of the infinite multi-point moment equations and PDF

  ⇒ Scaling laws can be derived rigorously from symmetries

  ⇒ Scaling laws are particularly simple in *H*-formulation

Thank you all for listening

It was a pleasure being here today

I am happy to take your question

TECHNISCHE UNIVERSITÄT DARMSTADT

UNIVERSITAT POLITÈCNICA DE VALÈNCIA

Key related publications:

1. Oberlack, Hoyas, et al.: PRL, **128**(2), 024502 (2022)
2. Hoyas, Oberlack, et al.: PRF, **7**(1), 014602 (2022)

All data are available @
https://doi.org/10.48328/tudatalib-670 and
https://doi.org/10.48328/tudatalib-658