Dynamics of the Atmosphere and the Ocean

Lecture 5

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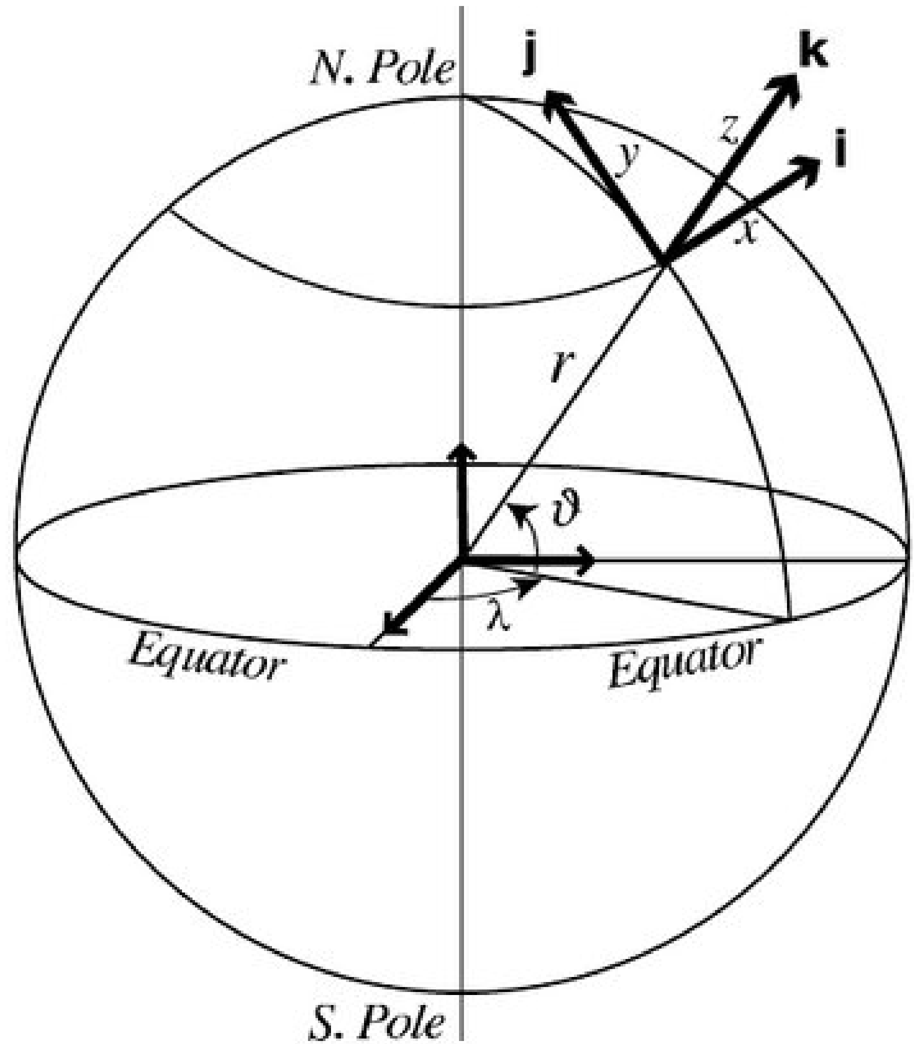
Figure 2.3 The spherical coordinate system. The orthogonal unit vectors **i**, **j** and **k** point in the direction of increasing longitude λ, latitude ϑ, and altitude z. Locally, one may apply a Cartesian system with variables x, y and z measuring distances along **i**, **j** and **k**.



Primitive equations

The inviscid momentum equation can be written as:

$$\frac{\mathrm{D}\boldsymbol{v}}{\mathrm{D}t} + 2\boldsymbol{\Omega} \times \boldsymbol{v} = -\frac{1}{\rho}\nabla p - \nabla\Phi.$$

(notice that here geopotential gradient $\vec{\nabla}\Phi = -\vec{g}$ stands for gravity acceleration)

This equation, together with conservation of mass (continuity equation) and conservation of energy (adiabatic, most often in the form:

$$\frac{\mathrm{D}\theta}{\mathrm{D}t} = 0)$$

are, after some approximations called “primitive equations” .

The typical approximations are:

1) the hydrostatic approximation: $$\frac{\partial p}{\partial z} = -\rho g.$$

2) the shallow fluid (shallow water) approximation: r=a+z (a- radius, z – height above sea level), and r is replaced by a everywhere except when differentiated:

$$\frac{1}{r^2}\frac{\partial(r^2 w)}{\partial r} \to \frac{\partial w}{\partial z}.$$

Let's consider momentum equation in Cartesian coordinates in a plane tangent to the surface of the Earth in a given location.

$$\frac{\partial u}{\partial t} + (\boldsymbol{v}\cdot\nabla)u + 2\Omega_y w - 2\Omega_z v = -\frac{1}{\rho}\frac{\partial p}{\partial x},$$

$$\frac{\partial v}{\partial t} + (\boldsymbol{v}\cdot\nabla)v + 2\Omega_z u = -\frac{1}{\rho}\frac{\partial p}{\partial y},$$

$$\frac{\partial w}{\partial t} + (\boldsymbol{v}\cdot\nabla)w + 2(\Omega_x v - \Omega_y u) = -\frac{1}{\rho}\frac{\partial p}{\partial z} - g,$$

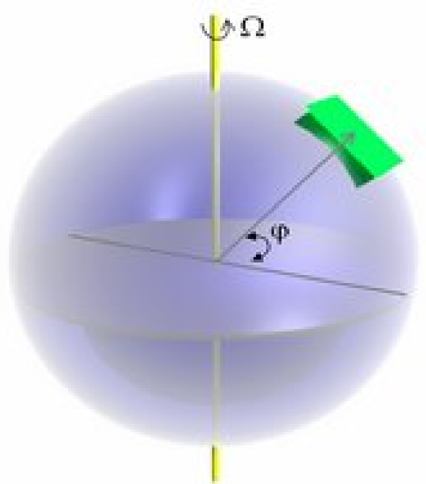
Here $\boldsymbol{\Omega} = \Omega_x\mathbf{i} + \Omega_y\mathbf{j} + \Omega_z\mathbf{k}$

If we ignore the components of Ω not in the direction of the local vertical, then

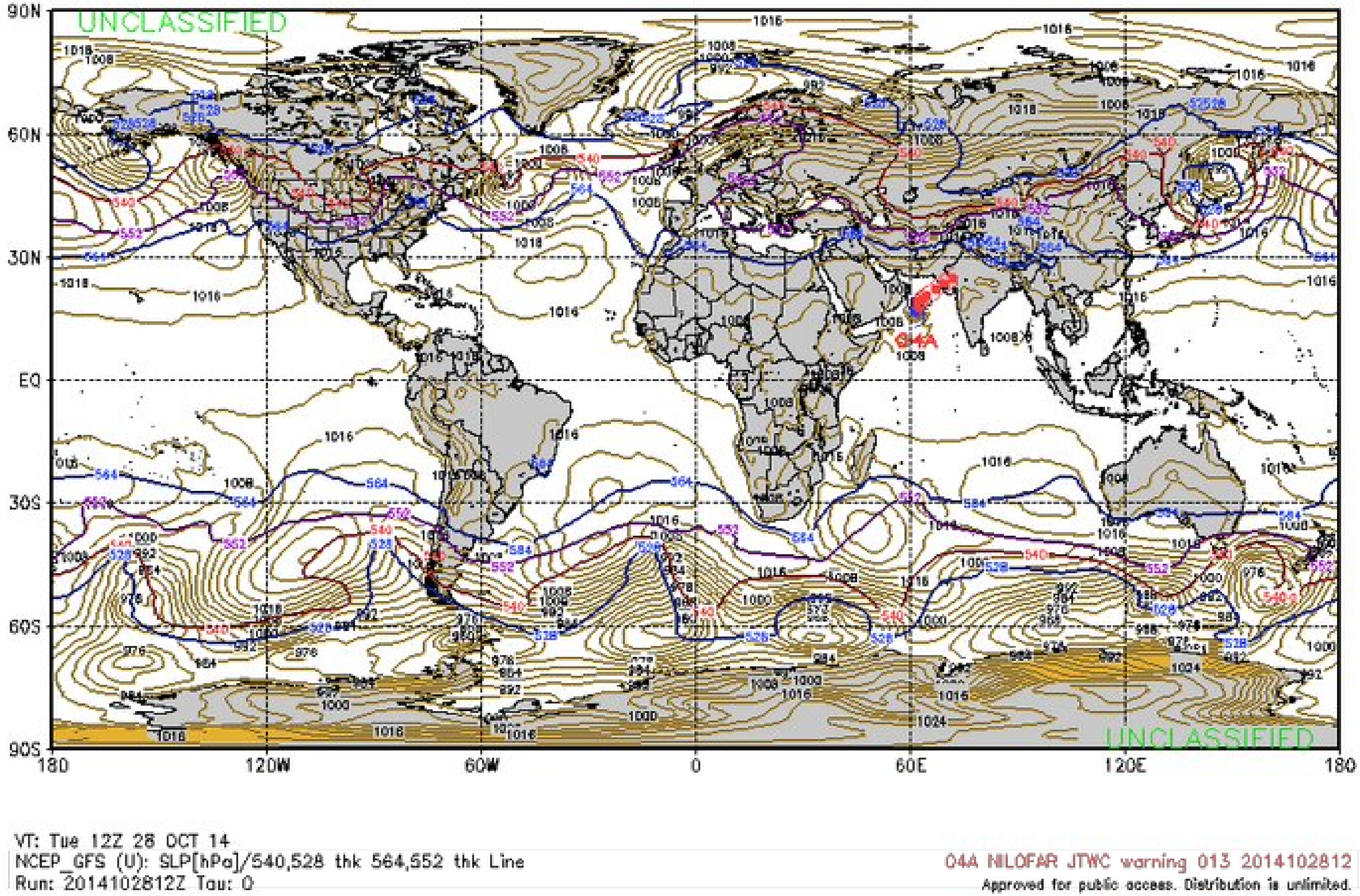
$$\frac{\mathrm{D}u}{\mathrm{D}t} - f_0 v = -\frac{1}{\rho}\frac{\partial p}{\partial x},$$

$$\frac{\mathrm{D}v}{\mathrm{D}t} + f_0 u = -\frac{1}{\rho}\frac{\partial p}{\partial y},$$

$$\frac{\mathrm{D}w}{\mathrm{D}t} = -\frac{1}{\rho}\frac{\partial p}{\partial z} - \rho g.$$



Here $f_0 = 2\Omega_z \sin\vartheta_0$ represents constant Coriolis parameter. The plane is tangent to the Earth's surface at the latitude ϑ_0. This approximation is called **„f-plane**”.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In reality, the magnitude of the vertical component of rotation varies with latitude, which is not accounted in f-plane.
One can approximate latitudinal variations by allowing the effective rotation vector to vary:

$$f = 2\Omega \sin\vartheta \approx 2\Omega \sin\vartheta_0 + 2\Omega \cos\vartheta_0 (\vartheta - \vartheta_0),$$

then on the tangent plane we may mimic this by allowing the Coriolis parameter to vary as

$$f = f_0 + \beta y$$

here $f_0 = 2\Omega \sin\vartheta_0$ and $\beta = \partial f/\partial y = (2\Omega \cos\vartheta_0)/a$

The above is known as the beta-plane approximation. It captures the the most important dynamical effects of sphericity, without the complicating geometric effects, which are not essential to describe many phenomena: f_0 is replaced by $f_0 + \beta y$ to represent a varying Coriolis parameter:

$$\frac{D\boldsymbol{u}}{Dt} + \boldsymbol{f} \times \boldsymbol{u} = -\frac{1}{\rho}\nabla_z p,$$

where $\boldsymbol{f} = (f_0 + \beta y)\hat{\mathbf{k}}$. In component form this equation becomes

$$\frac{Du}{Dt} - fv = -\frac{1}{\rho}\frac{\partial p}{\partial x}, \qquad \frac{Dv}{Dt} + fu = -\frac{1}{\rho}\frac{\partial p}{\partial y},$$

THE BOUSSINESQ APPROXIMATION

The density variations in the ocean and in the atmospheric boundary layer are quite small compared to the mean density. In the ocean three effects are important: the compression of water by pressure (p), the thermal expansion of water if its temperature changes (T), and the haline contraction if its salinity changes (S). An appropriate equation of state to approximately evaluate these effects is the linear one:

$$\rho = \rho_0 \left[1 - \beta_T (T - T_0) + \beta_S (S - S_0) + \frac{p}{\rho_0 c_s^2} \right],$$

$$\beta_T \approx 2 \times 10^{-4}\,\mathrm{K}^{-1},\ \beta_S \approx 10^{-3}\,\mathrm{psu}^{-1} \text{ and } c_s \approx 1500\,\mathrm{m\,s}^{-1}$$

In 1978, oceanographers redefined salinity in the Practical Salinity Scale (PSS) as the conductivity ratio of a sea water sample to a standard KCl solution. Although PSS is a dimensionless quantity, its "unit" is usually called PSU. Salinity of 35 equals 35 grams of salt per liter of solution.

Pressure compressibility: $\Delta_p \rho \approx \Delta p / c^2 \approx \rho_0 g H / c^2$ $\quad \frac{|\Delta_p \rho|}{\rho_0} \ll 1 \quad \text{if} \quad \frac{gH}{c^2} \ll 1,$

Thermal expansion: $\Delta_T \rho \approx -\beta_T \rho_0 \Delta T$ $\quad \frac{|\Delta_T \rho|}{\rho_0} \ll 1 \quad \text{if} \quad \beta_T \Delta T \ll 1.$

Saline contraction: $\Delta_S \rho \approx \beta_S \rho_0 \Delta S$ $\quad \frac{|\Delta_S \rho|}{\rho_0} \ll 1 \quad \text{if} \quad \beta_S \Delta S \ll 1.$

i.e. In the ocean density fluctuations are small.

The Boussinesq equations are a set of equations that exploit the smallness of density Variations. We may write:

$$\begin{aligned}\rho &= \rho_0 + \delta\rho(x, y, z, t)\\ &= \rho_0 + \widehat{\rho}(z) + \rho'(x, y, z, t)\\ &= \widetilde{\rho}(z) + \rho'(x, y, z, t)\end{aligned}$$

and

$$|\widehat{\rho}|, |\rho'|, |\delta\rho| \ll \rho_0.$$

or Boussinesq approximation.

Associated with the reference density is a reference pressure in hydrostatic balance with it:

$$\begin{aligned}p &= p_0(z) + \delta p(x, y, z, t)\\ &= \widetilde{p}(z) + p'(x, y, z, t),\end{aligned}$$

where $|\delta p| \ll p_0$, $|p'| \ll \widetilde{p}$ and

$$\frac{\mathrm{d}p_0}{\mathrm{d}z} \equiv -g\rho_0, \qquad \frac{\mathrm{d}\widetilde{p}}{\mathrm{d}z} \equiv -g\widetilde{\rho}.$$

Note that $\nabla_z p = \nabla_z p' = \nabla_z \delta p$ and that $p_0 \approx \widetilde{p}$ if $|\widehat{\rho}| \ll \rho_0$.

gradient in horizontal

The momentum equation can be written (note horizontal gradient operator without z) as:

$$(\rho_0 + \delta\rho)\left(\frac{\mathrm{D}\boldsymbol{v}}{\mathrm{D}t} + 2\boldsymbol{\Omega} \times \boldsymbol{v}\right) = -\nabla\delta p - \frac{\partial p_0}{\partial z}\mathbf{k} - g(\rho_0 + \delta\rho)\mathbf{k},$$

Accounting for hydrostatic balance of the reference density and pressure we get:

$$(\rho_0 + \delta\rho)\left(\frac{\mathrm{D}\boldsymbol{v}}{\mathrm{D}t} + 2\boldsymbol{\Omega} \times \boldsymbol{v}\right) = -\nabla\delta p - g\delta\rho\mathbf{k}.$$

For small differences of density

$$\left(\frac{\mathrm{D}\boldsymbol{v}}{\mathrm{D}t} + 2\boldsymbol{\Omega} \times \boldsymbol{v}\right) = -\nabla\phi + b\mathbf{k}$$

Where

$$\phi = \delta p/\rho_0 \qquad\qquad b = -g\delta\rho/\rho_0$$

and b stays for buoyancy.
It is common to say that the Boussinesq approximation ignores all variations of density of a fluid in the momentum equation, except when associated with the gravitational term.

Typically for most large-scale motions the deviation pressure and density fields are also approximately in hydrostatic balance,which results in:

$$\frac{\partial \phi}{\partial z} = b.$$

A condition for the above to hold is that vertical accelerations are small compared to $g\delta\rho / \rho_0$, and not compared to the acceleration due to gravity itself.

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \boldsymbol{v}) = 0$$

Mass conservation:

$$\frac{D\delta\rho}{Dt} + (\rho_0 + \delta\rho)\nabla \cdot \boldsymbol{v} = 0.$$

When the total derivative in the above and advection scale in the same way then the above can be approximated as:

$$\nabla \cdot \boldsymbol{v} = 0$$

Note that the evolution of density (leftmost term) does NOT follow from the above momentum equations!!!
It is given by the thermodynamic equation in conjunction with an equation of state, and this should not be confused with the mass conservation equation.

Thermodynamic equation

Neglecting salinity a useful starting point is to write the thermodynamic equation as:

$$\frac{D\rho}{Dt} - \frac{1}{c^2}\frac{Dp}{Dt} = \frac{\dot{Q}}{(\partial\eta/\partial\rho)_p T} \approx -\dot{Q}\left(\frac{\rho_0\beta_T}{c_p}\right)$$

using $(\partial\eta/\partial\rho)_p = (\partial\eta/\partial T)_p(\partial T/\partial\rho)_p \approx c_p/(T\rho_0\beta_T)$.

Then, from Boussinesq approximations for density and pressure one may write:

$$\frac{D\delta\rho}{Dt} - \frac{1}{c^2}\frac{Dp_0}{Dt} = -\dot{Q}\left(\frac{\rho_0\beta_T}{c_p}\right),$$

$$\frac{D}{Dt}\left(\delta\rho + \frac{\rho_0 g}{c^2}z\right) = -\dot{Q}\left(\frac{\rho_0\beta_T}{c_p}\right),$$

The above, keeping in mind that $b = -g\delta\rho/\rho_0$ can be approximated as:

$$\frac{Db}{Dt} = \dot{b} \qquad \dot{b} = g\beta_T\dot{Q}/c_p.$$

The above set of equations (momentum, mass continuity equation and thermodynamic equation) form a closed set, called the simple Boussinesq equations.

The Boussinesq equations (in a slightly more general form, using equation of state – look into Valis book), with the hydrostatic and traditional approximations are often considered to be the oceanic primitive,equations:

Summary of Boussinesq Equations

The simple Boussinesq equations are, for an inviscid fluid:

Momentum equations: $$\frac{D\boldsymbol{v}}{Dt} + \boldsymbol{f} \times \boldsymbol{v} = -\nabla\phi + b\mathbf{k} \qquad \text{(B.1)}$$

Mass conservation: $$\nabla \cdot \boldsymbol{v} = 0 \qquad \text{(B.2)}$$

Buoyancy equation: $$\frac{Db}{Dt} = \dot{b} \qquad \text{(B.3)}$$

A more general form replaces the buoyancy equation by:

Thermodynamic equation: $$\frac{D\theta}{Dt} = \dot{\theta} \qquad \text{(B.4)}$$

Salinity equation: $$\frac{DS}{Dt} = \dot{S} \qquad \text{(B.5)}$$

Equation of state: $$b = b(\theta, S, z) \qquad \text{(B.6)}$$

Energetics of the Boussinesq system

In a uniform gravitational field, with no other forcing or dissipation, the Boussinesq equations are:

$$\frac{D\boldsymbol{v}}{Dt} + 2\boldsymbol{\Omega} \times \boldsymbol{v} = b\mathbf{k} - \nabla\phi, \qquad \nabla \cdot \boldsymbol{v} = 0, \qquad \frac{Db}{Dt} = 0.$$

Taking dot product of the momentum equation with v we obtain an equation for the evolution of kinetic energy density:

$$\frac{1}{2}\frac{Dv^2}{Dt} = bw - \nabla \cdot (\phi \boldsymbol{v})$$

Taking $\nabla\Phi = -\mathbf{k}$ (so $\Phi = -z$) and differentiating one gets:

$$\frac{D\Phi}{Dt} = \nabla \cdot (\boldsymbol{v}\Phi) = -w.$$

The above, together with results in the equation for the evolution of potential energy:

$$\frac{D}{Dt}(b\Phi) = -wb.$$

Adding potential and kinetic energies and expanding the material derivative one obtains an energy equation for the Boussinesq system:

$$\frac{\partial}{\partial t}\left(\frac{1}{2}\boldsymbol{v}^2 + b\Phi\right) + \nabla \cdot \left[\boldsymbol{v}\left(\frac{1}{2}\boldsymbol{v}^2 + b\Phi + \phi\right)\right] = 0.$$

The energy density (divided by ρ_0) is just $\boldsymbol{v}^2/2 + b\Phi$. .
Integral of the second term multiplied by ρ_0 is the potential energy of the flow minus that of the basic state.
If there were a heating term on the right-hand side of $\frac{\mathrm{D}b}{\mathrm{D}t} = 0.$
this would directly provide a source of
potential energy, rather than internal energy as in the compressible system.

Because the fluid is incompressible, there is no conversion from kinetic and potential energy into internal energy.

A liquid ocean

A sometimes-useful expression for stability arises by noting that in an adiabatic displacement

$$\delta\rho_\theta = \delta\rho - \frac{1}{c_s^2}\delta p = 0. \tag{2.224}$$

If the fluid is hydrostatic $\delta p = -\rho g \delta z$ so that if a parcel is displaced adiabatically its density changes according to

$$\left(\frac{\partial \rho}{\partial z}\right)_{\rho_\theta} = -\frac{\rho g}{c_s^2}. \tag{2.225}$$

If a parcel is displaced a distance δz upwards then the density difference between it and its new surroundings is

$$\delta\rho = -\left[\left(\frac{\partial \rho}{\partial z}\right)_{\rho_\theta} - \left(\frac{\partial \widetilde{\rho}}{\partial z}\right)\right]\delta z = \left[\frac{\rho g}{c_s^2} + \left(\frac{\partial \widetilde{\rho}}{\partial z}\right)\right]\delta z, \tag{2.226}$$

which gives:

$$N^2 = -g\left[\frac{g}{c_s^2} + \frac{1}{\widetilde{\rho}}\left(\frac{\partial \widetilde{\rho}}{\partial z}\right)\right]$$

2.10.1 Gravity waves and convection in a Boussinesq fluid

Let us consider a Boussineq fluid, at rest, in which the buoyancy varies linearly with height and the bouyancy frequency, N, is a constant. Linearizing the equations of motion about this basic state we obtain

$$\frac{\partial u'}{\partial t} = -\frac{\partial \phi'}{\partial x}, \tag{2.243a}$$

$$\frac{\partial w'}{\partial t} = -\frac{\partial \phi'}{\partial z} + b', \tag{2.243b}$$

$$\frac{\partial u'}{\partial x} + \frac{\partial w'}{\partial z} = 0, \tag{2.243c}$$

$$\frac{\partial b'}{\partial t} + w' N^2 = 0, \tag{2.243d}$$

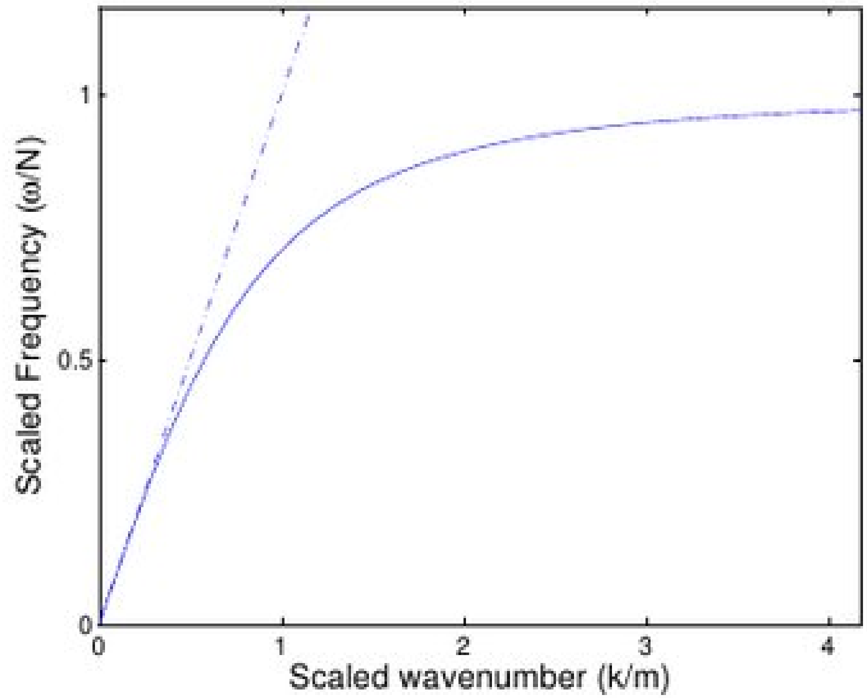
where for simplicity we assume that the flow is a function only of x and z. A little algebra gives a single equation for w',

$$\left[\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial z^2}\right)\frac{\partial^2}{\partial t^2} + N^2 \frac{\partial^2}{\partial x^2}\right] w' = 0. \tag{2.244}$$

Seeking solutions of the form $w' = \mathrm{Re}\, W \exp[i(kx + mz - \omega t)]$ (where Re denotes the real part) yields the important dispersion relationship for gravity waves:

$$\boxed{\omega^2 = \frac{k^2 N^2}{k^2 + m^2}}. \tag{2.245}$$

Figure 2.7 Scaled frequency, ω/N, plotted as a function of scaled horizontal wavenumber, k/m, using the full dispersion relation of (2.245) (solid line, asymptoting to unit value for large k/m) and with the hydrostatic dispersion relation (2.249) (dashed line, tending to ∞ for large k/m).



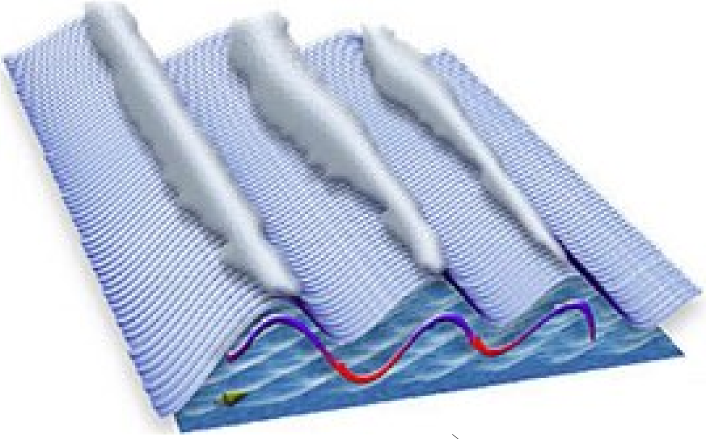
The frequency (see Fig. 2.7) is thus always less than N, approaching N for small horizontal scales, $k \gg m$. If we explicitly neglect pressure perturbations, as in the parcel argument, then the two equations,

$$\frac{\partial w'}{\partial t} = b', \qquad \frac{\partial b'}{\partial t} + w' N^2 = 0, \tag{2.246}$$

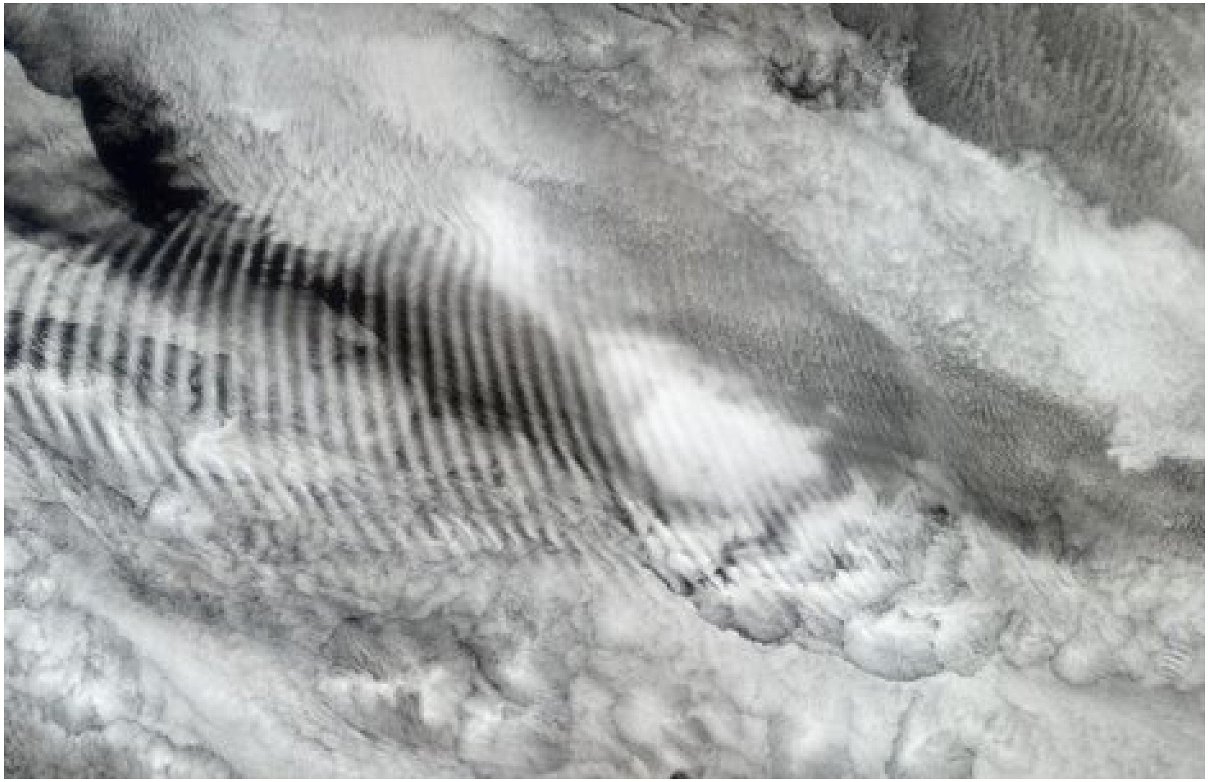
form a closed set, and give $\omega^2 = N^2$.

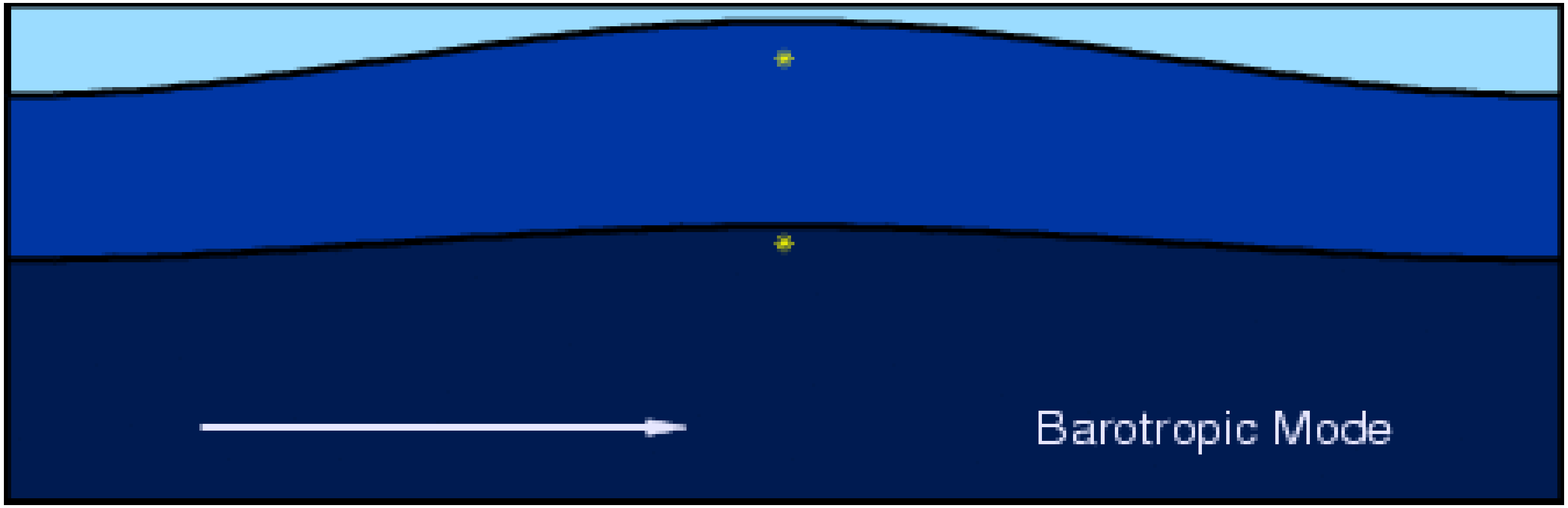
If the basic state density increases with height then $N^2 < 0$ and we expect this state to be unstable. Indeed, (2.245) then gives

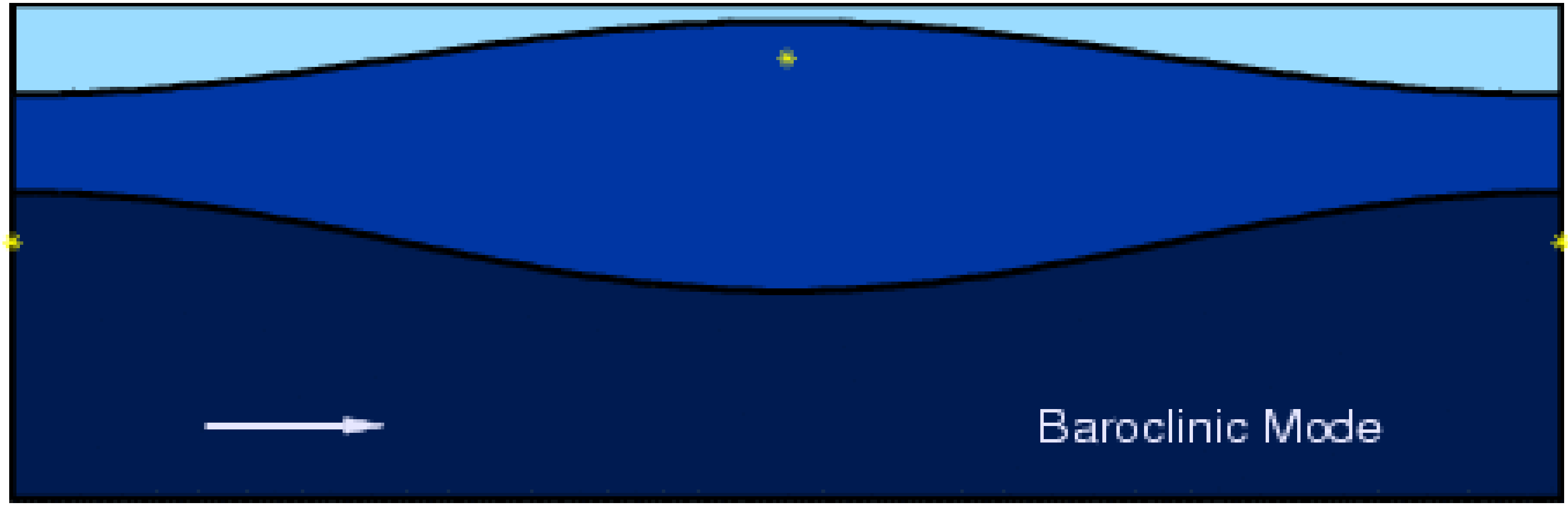
$$\sigma = \frac{\pm k \widetilde{N}}{(k^2 + m^2)^{1/2}}, \tag{2.247}$$



Example of gravity waves in the atmosphere visualized by condensation in the wave crest.

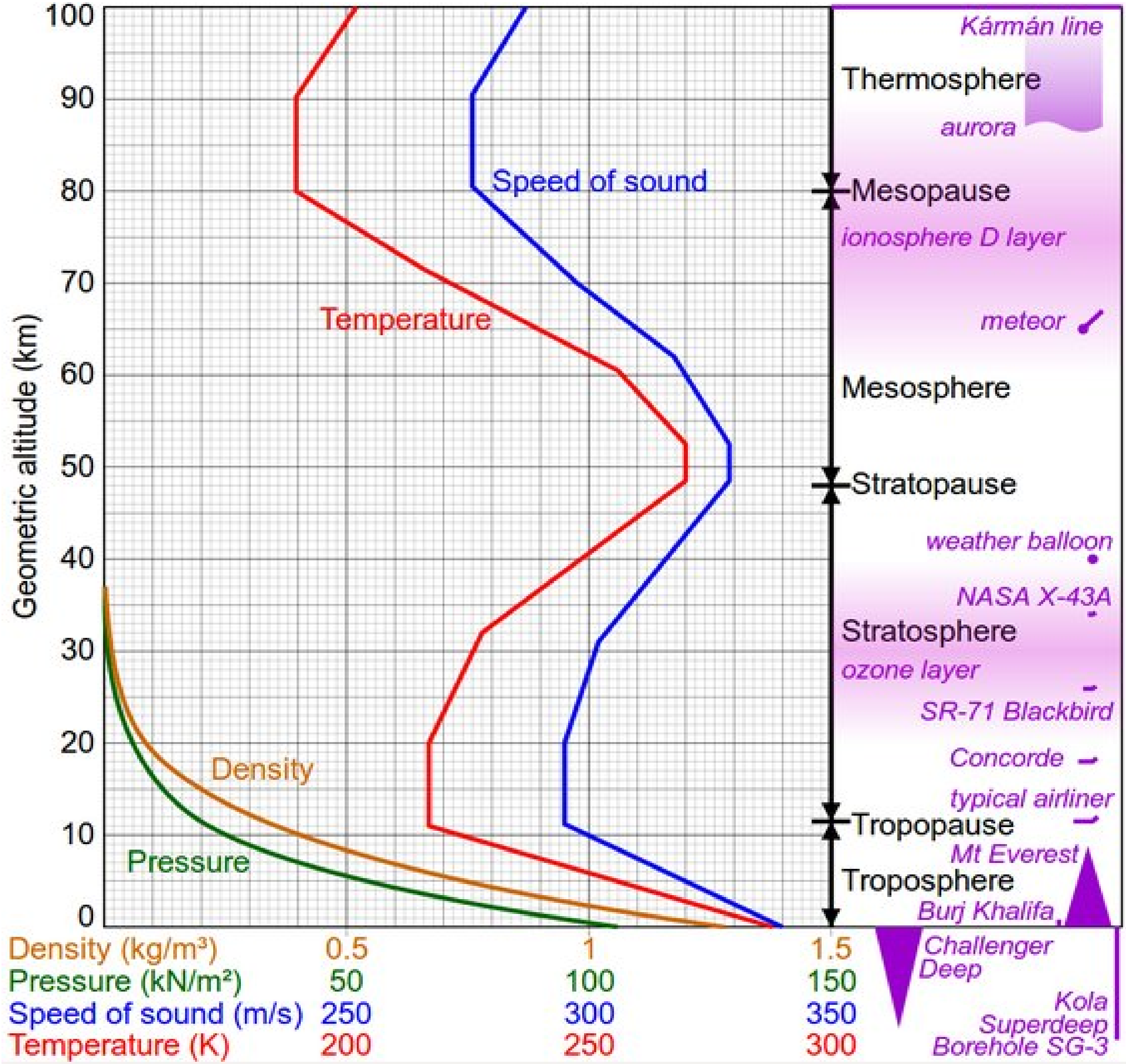




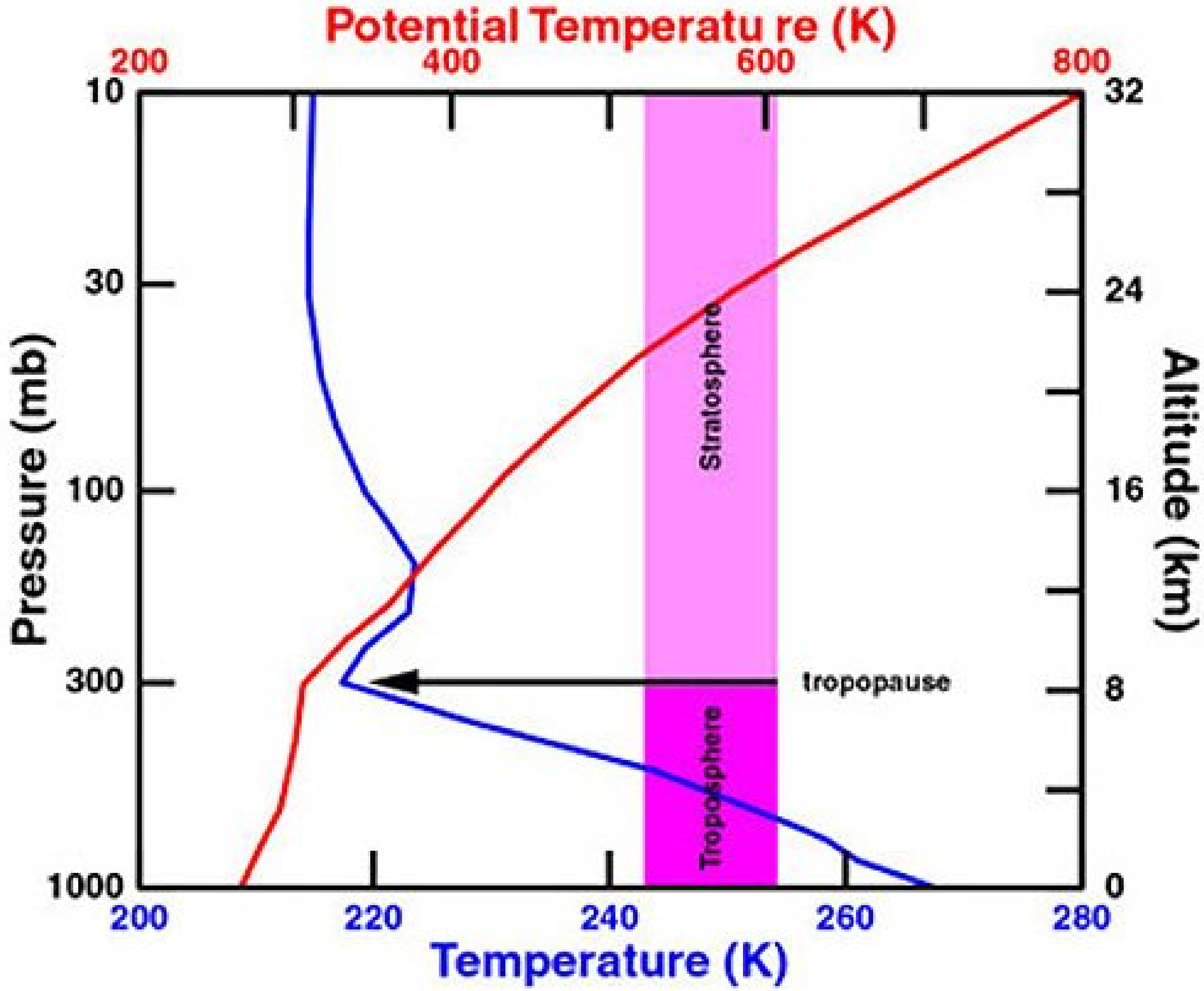


From: http://www.student.math.uwaterloo.ca/~amat361/Fluid%20Mechanics/topics/internal_waves.htm

for future...



Standard atmosphere



February 24, 1999 75°W, 40 °N

EQUATIONS FOR A STRATIFIED ATMOSPHERE: THE ANELASTIC APPROXIMATION

In the atmosphere the density varies significantly, mainly in the vertical. Deviations of both density and pressure from a statically balanced state are often small, the relative vertical variation of potential temperature is also small. This allows to formulate simplified set of equations, useful for theoretical and numerical analysis because sound waves are eliminated by way of an ‘anelastic’ approximation. To begin we set:

$$\rho = \widetilde{\rho}(z) + \delta\rho(x, y, z, t), \qquad p = \widetilde{p}(z) + \delta p(x, y, z, t)$$

$$|\delta\rho| \ll |\widetilde{\rho}|$$

$$\frac{\partial \widetilde{p}}{\partial z} \equiv -g\widetilde{\rho}(z).$$

Importantly, the density basic state is now a (given) function of vertical coordinate. As with the Boussinesq, the idea is to ignore dynamic variations of density except where associated with gravity.

Remember that, air can be considered an ideal gas fulfilling:

$$s \equiv \log\theta = \log T - \frac{R}{c_p}\log p = \frac{1}{\gamma}\log p - \log\rho,$$

$\gamma = c_p/c_v$

$$\delta s = \frac{1}{\gamma}\frac{\delta p}{p} - \frac{\delta\rho}{\rho} \approx \frac{1}{\gamma}\frac{\delta p}{\widetilde{p}} - \frac{\delta\rho}{\widetilde{\rho}}$$

When:

$$\tilde{s} \equiv \gamma^{-1} \log \tilde{p} - \log \tilde{\rho}$$

Then

$$\frac{\mathrm{d}\tilde{s}}{\mathrm{d}z} = \frac{1}{\gamma \tilde{p}} \frac{\mathrm{d}\tilde{p}}{\mathrm{d}z} - \frac{1}{\tilde{\rho}} \frac{\mathrm{d}\tilde{\rho}}{\mathrm{d}z} = -\frac{g\tilde{\rho}}{\gamma \tilde{p}} - \frac{1}{\tilde{\rho}} \frac{\mathrm{d}\tilde{\rho}}{\mathrm{d}z}.$$

In the atmosphere, the left-hand side is, typically, much smaller than either of the two terms on the right-hand side.

The (horizontal) momentum equation

$$(\tilde{\rho} + \rho') \frac{\mathrm{D}\boldsymbol{u}}{\mathrm{D}t} + \boldsymbol{f} \times \boldsymbol{u} = -\nabla_z \delta p.$$

Neglecting density fluctuations we get:

$$\frac{\mathrm{D}\boldsymbol{u}}{\mathrm{D}t} + \boldsymbol{f} \times \boldsymbol{u} = -\nabla_z \phi,$$

Where

$$\phi = \delta p / \tilde{\rho},$$

The above is similar to the corresponding equation in the Boussinesq approximation.

Consider vertical component of the momentum equation (using decomposition of pressure and density into reference and fluctuating parts):

$$(\widetilde{\rho}+\delta\rho)\frac{\mathrm{D}w}{\mathrm{D}t} = -\frac{\partial\widetilde{p}}{\partial z} - \frac{\partial\delta p}{\partial z} - g\widetilde{\rho} - g\delta\rho = -\frac{\partial\delta p}{\partial z} - g\delta\rho.$$

Neglecting $\delta\rho$ on the left-hand-side we obtain:

$$\frac{\mathrm{D}w}{\mathrm{D}t} = -\frac{1}{\widetilde{\rho}}\frac{\partial\delta p}{\partial z} - g\frac{\delta\rho}{\widetilde{\rho}} = -\frac{\partial}{\partial z}\left(\frac{\delta p}{\widetilde{\rho}}\right) - \frac{\delta p}{\widetilde{\rho}^2}\frac{\partial\widetilde{\rho}}{\partial z} - g\frac{\delta\rho}{\widetilde{\rho}}.$$

Now we have to eliminate $\delta\rho$ in favor of δs:

$$\frac{\mathrm{D}w}{\mathrm{D}t} = g\delta s - \frac{\partial}{\partial z}\left(\frac{\delta p}{\widetilde{\rho}}\right) - \frac{g}{\gamma}\frac{\delta p}{\widetilde{p}} - \frac{\delta p}{\widetilde{\rho}^2}\frac{\partial\widetilde{\rho}}{\partial z},$$

$$\frac{\mathrm{D}w}{\mathrm{D}t} = g\delta s - \frac{\partial}{\partial z}\left(\frac{\delta p}{\widetilde{\rho}}\right) + \frac{\mathrm{d}\widetilde{s}}{\mathrm{d}z}\frac{\delta p}{\widetilde{\rho}}.$$

(i) The gravitational term now involves δs rather than $\delta\rho$ which enables a more direct connection with the thermodynamic equation.
(ii) The potential temperature scale height (100 km) in the atmosphere is much larger than the density scale height (10 km), and so the last term in the above is small.

When we choose the reference state to be one of constant potential temperature the term $\mathrm{d}\tilde{s}/\mathrm{d}z$ vanishes and the vertical momentum equation becomes:

$$\frac{\mathrm{D}w}{\mathrm{D}t} = g\,\delta s - \frac{\partial \phi}{\partial z}$$

$$\phi = \delta p/\tilde{\rho},\ \delta s = \delta\theta/\tilde{\theta} \text{ and } \tilde{\theta} = \theta_0.$$

We have now the same form as the Boussinesq momentum equations, but with different definitions of geopotential (above) and buoyancy:

$$b_a \equiv g\,\delta s = g\,\delta\theta/\tilde{\theta}$$

Mass conservation:

$$\frac{\partial \delta\rho}{\partial t} + \nabla\cdot[(\tilde{\rho} + \delta\rho)\boldsymbol{v}] = 0.$$

We neglect $\delta\rho$ in the divergence term. Further, the local time derivative will be small if time itself is scaled advectively (i.e., T/L=U and sound waves do not dominate). This allows to replace temporal derivative of density fluctuations into rate of change of the reference density in vertical motions, giving:

$$\nabla\cdot\boldsymbol{u} + \frac{1}{\tilde{\rho}}\frac{\partial}{\partial z}(\tilde{\rho}w) = 0$$

It is here that the eponymous ‘anelastic approximation’ arises: the elastic compressibility of the fluid is neglected, and this serves to eliminate sound waves. For reference, in spherical coordinates the mass conservation equation is:

$$\frac{1}{a\cos\vartheta}\frac{\partial u}{\partial \lambda}+\frac{1}{a\cos\vartheta}\frac{\partial}{\partial\vartheta}(v\cos\vartheta)+\frac{1}{\widetilde{\rho}}\frac{\partial(w\widetilde{\rho})}{\partial z}=0.$$

Thermodynamic equation:

$$\frac{\mathrm{D}\ln\theta}{\mathrm{D}t}=\frac{\dot{Q}}{Tc_p}.$$

Can be approximated as:

$$\frac{\mathrm{D}\delta s}{\mathrm{D}t}=\frac{\widetilde{\theta}}{Tc_p}\dot{Q}.$$

And the whole anelastic approximation for Adiabatic, non viscous flow is:

$$b_a = g\delta s = g\delta\theta/\widetilde{\theta}.$$

$$\boxed{\begin{gathered}\frac{\mathrm{D}\boldsymbol{v}}{\mathrm{D}t}+2\boldsymbol{\Omega}\times\boldsymbol{v}=\mathbf{k}b_a-\nabla\phi\\ \frac{\mathrm{D}b_a}{\mathrm{D}t}=0\\ \nabla\cdot(\widetilde{\rho}\boldsymbol{v})=0\end{gathered}}$$

Journal of the Atmospheric Sciences
Volume 19, Issue 2 (March 1962) pp. 173-179

Scale Analysis of Deep and Shallow Convection in the Atmosphere[1]

YOSHIMITSU OGURA AND NORMAN A. PHILLIPS
Massachusetts Institute of Technology

(Manuscript received 18 October 1961, in revised form 27 November 1961)

ABSTRACT

The approximate equations of motion derived by Batchelor in 1953 are derived by a formal scale analysis, with the assumption that the percentage range in potential temperature is small and that the time scale is set by the Brunt-Väisälä frequency. Acoustic waves are then absent. If the vertical scale is small compared to the depth of an adiabatic atmosphere, the system reduces to the (non-viscous) Boussinesq equations. The computation of the saturation vapor pressure for *deep* convection is complicated by the important effect of the dynamic pressure on the temperature. For *shallow* convection this effect is not important, and a simple set of reversible equations is derived.

Energetics of the anelastic equations

Kinetic energy equation, obtained in the same manner as in Boussinesq case is:

$$\tilde{\rho}\frac{\mathrm{D}}{\mathrm{D}t}\left(\frac{1}{2}\boldsymbol{v}^2\right) = -\nabla\cdot(\phi\tilde{\rho}\boldsymbol{v}) + b_a\tilde{\rho}w$$

Define geopotential $\Phi(z)$ such that $\nabla\Phi = -\boldsymbol{k}$ **:**

$$\tilde{\rho}\frac{\mathrm{D}\Phi}{\mathrm{D}t} = -w\tilde{\rho}.$$

Taking into account

$$\frac{\mathrm{D}b_a}{\mathrm{D}t} = 0$$

one gets equation for the rate of change of potential energy:

$$\tilde{\rho}\frac{\mathrm{D}(b_a\Phi)}{\mathrm{D}t} = -wb_a\tilde{\rho}.$$

Combining it with the kinetic energy equation results in:

$$\tilde{\rho}\frac{\mathrm{D}}{\mathrm{D}t}\left(\frac{1}{2}\boldsymbol{v}^2 + b_a\Phi\right) = -\nabla\cdot(\phi\tilde{\rho}\boldsymbol{v}),$$

After expanding the material derivative and rearrangement

$$\frac{\partial}{\partial t}\left[\tilde{\rho}\left(\frac{1}{2}\boldsymbol{v}^2 + b_a\Phi\right)\right] + \nabla\cdot\left[\tilde{\rho}\boldsymbol{v}\left(\frac{1}{2}\boldsymbol{v}^2 + b_a\Phi + \phi\right)\right] = 0.$$

This can be written as:

$$\frac{\partial E}{\partial t} + \nabla\cdot[\boldsymbol{v}(E + \tilde{\rho}\phi)] = 0$$

$$E = \tilde{\rho}(\boldsymbol{v}^2/2 + b_a\Phi)$$

Where E is the total energy of the flow. Energy, when integrated over the closed domain (whatever it means) is conserved. Term $\tilde{\rho}b_a\Phi$ is analogous to the potential energy of a Boussinesq system, but exactly equal to that because b_a is the bouyancy based on potential temperature, not density. The term combines contributions from both the internal energy and the potential energy.