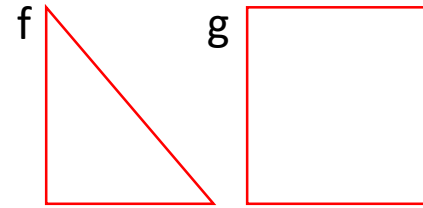
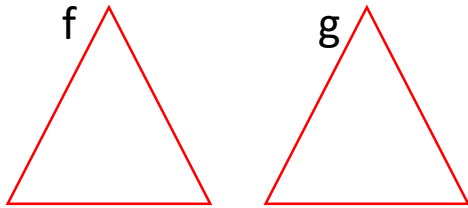


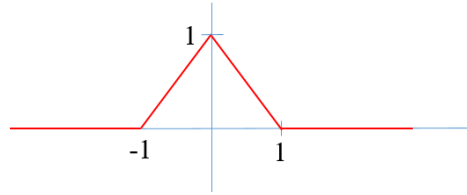
1100-4BW12, rok akademicki 2018/19

WSTĘP DO OPTYKI FOURIEROWSKIEJ

dr hab. Rafał Kasztelanic



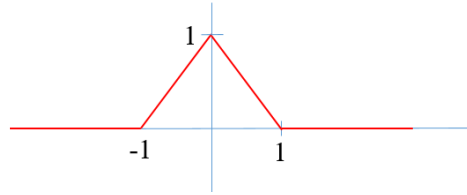
$$\Lambda(x) = \begin{cases} 1 - |x| & |x| < 1 \\ 0 & |x| \geq 1 \end{cases}$$



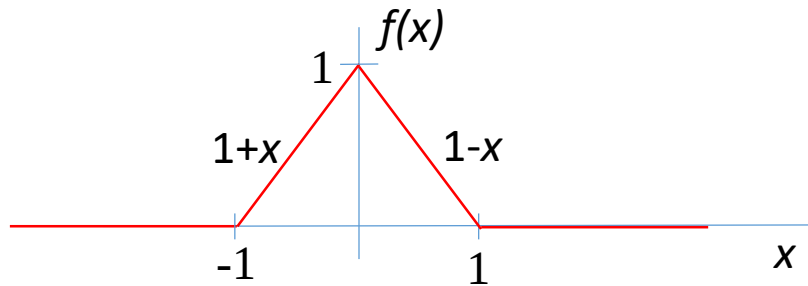
$$h(x) = \int_{-\infty}^{\infty} f(x-x') g(x') dx'$$

Splot

$$\Lambda(x) = \begin{cases} 1 - |x| & |x| < 1 \\ 0 & |x| \geq 1 \end{cases}$$

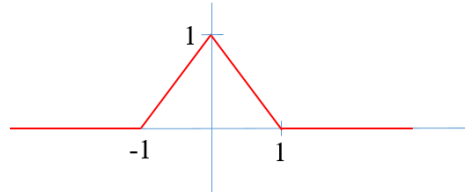


$$h(x) = \int_{-\infty}^{\infty} f(x - x') g(x') dx'$$

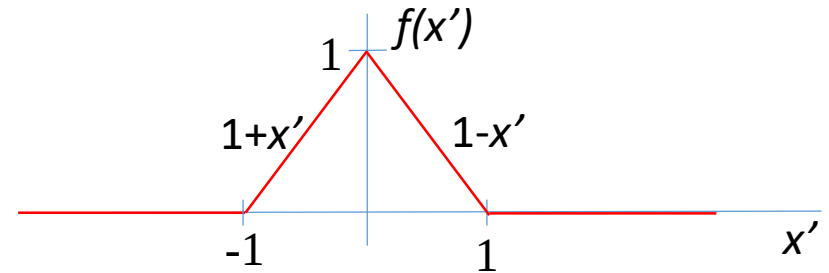
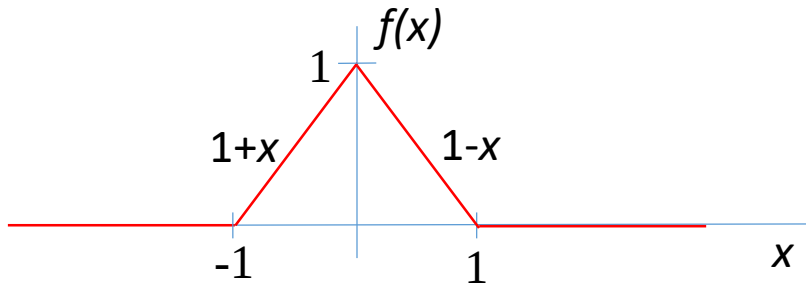


Splot

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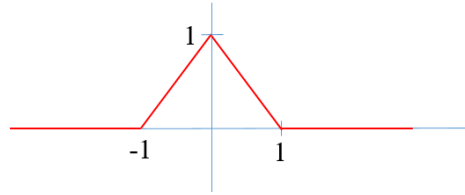


$$h(x) = \int_{-\infty}^{\infty} f(x - x') g(x') dx'$$

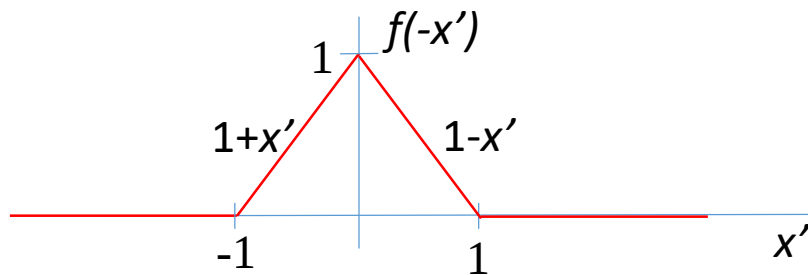
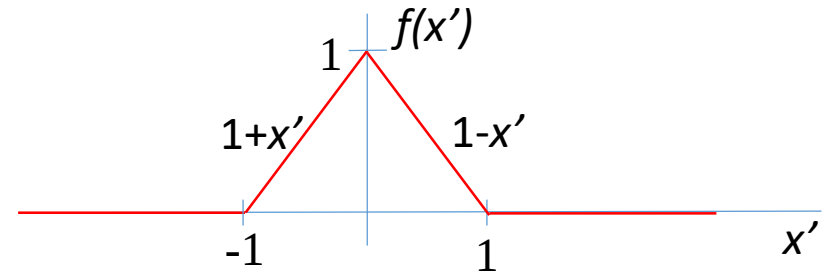
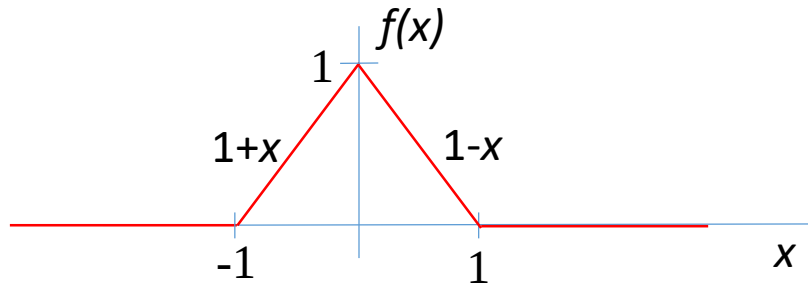


Splot

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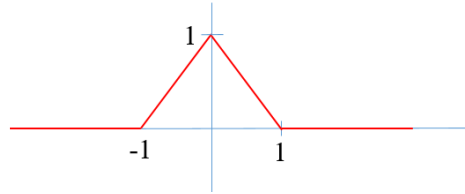


$$h(x) = \int_{-\infty}^{\infty} f(x - x') g(x') dx'$$

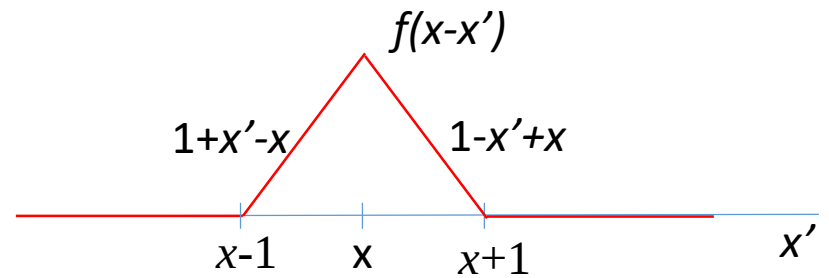
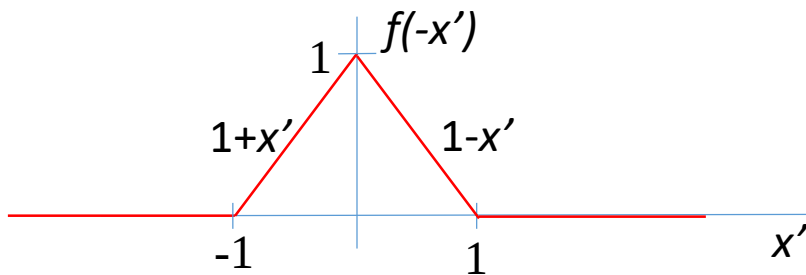
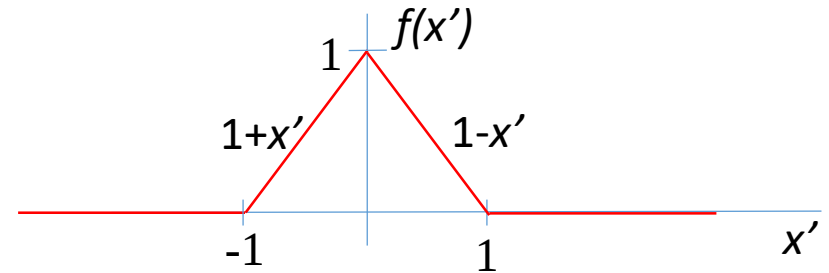
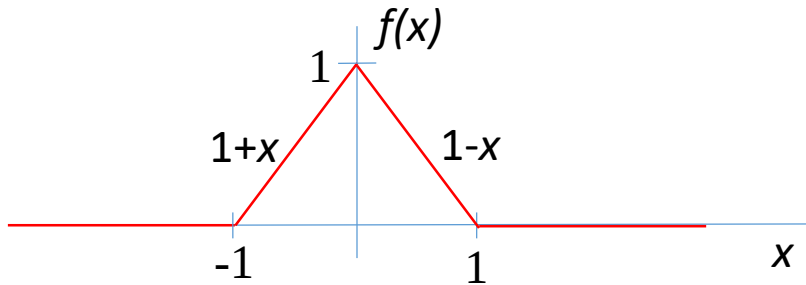


Splot

$$\Lambda(x) = \begin{cases} 1 - |x| & |x| < 1 \\ 0 & |x| \geq 1 \end{cases}$$

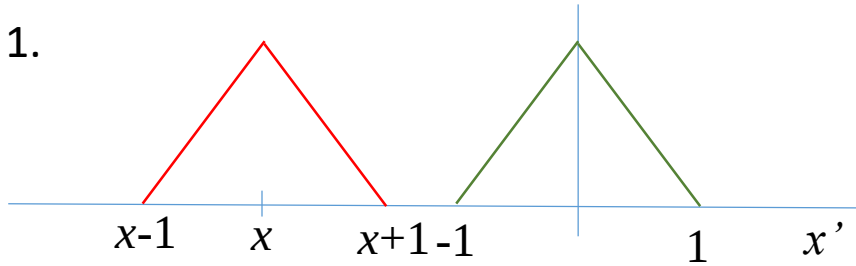


$$h(x) = \int_{-\infty}^{\infty} f(x - x') g(x') dx'$$



Splot

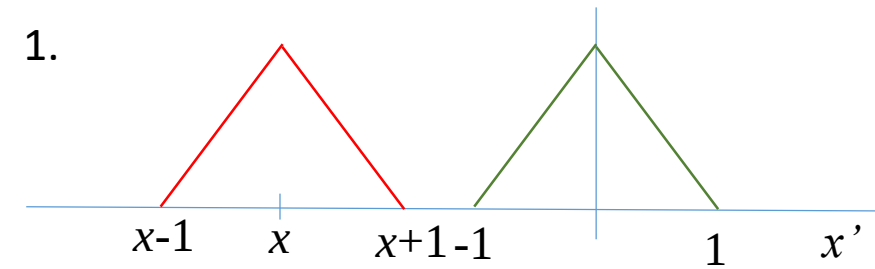
1.



$$x + 1 < -1 \rightarrow x < -2$$

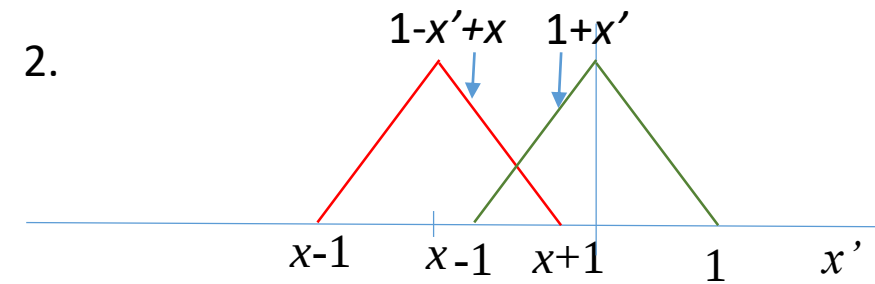
$$h(x) = 0$$

Splot



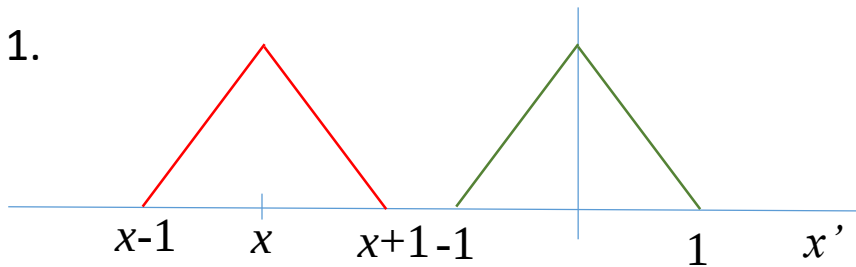
$$x + 1 < -1 \rightarrow x < -2$$

$$h(x) = 0$$



$$-1 < x + 1 < 0 \rightarrow -2 < x < -1$$

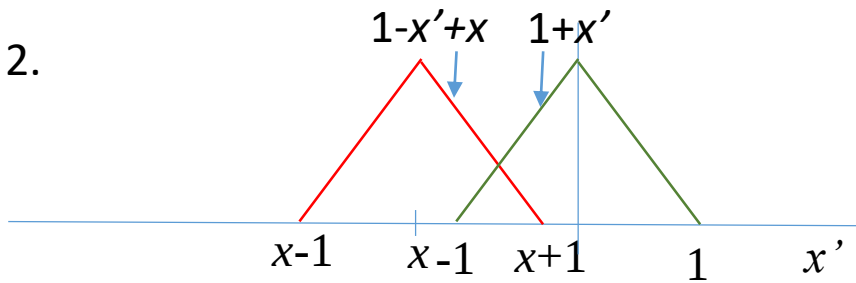
1.



$$x + 1 < -1 \rightarrow x < -2$$

$$h(x) = 0$$

2.



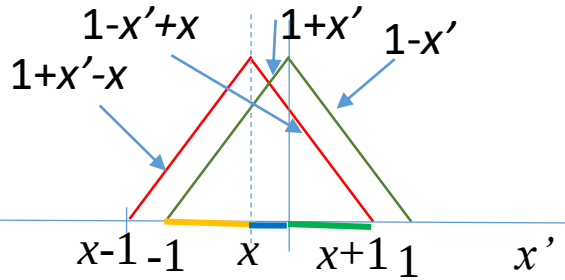
$$-1 < x + 1 < 0 \rightarrow -2 < x < -1$$

$$h(x) = \int_{-1}^{x+1} (1 + x')(1 - x' + x) dx' =$$

$$= \int_{-1}^{x+1} (1 + x - x'^2 + xx') dx' = \underbrace{(1 + x) \int_{-1}^{x+1} dx'}_{-(x+1)(x+2)} + \underbrace{x \int_{-1}^{x+1} x' dx'}_{-\frac{x}{2}((x+1)^2 - 1)} - \underbrace{\int_{-1}^{x+1} x'^2 dx'}_{-\frac{1}{3} - \frac{(x+1)^3}{3}} =$$

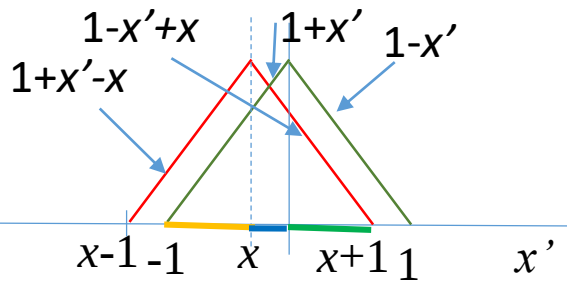
$$h(x) = \frac{1}{6}(x+2)^3$$

3.



$$0 < x + 1 < 1 \rightarrow -1 < x < 0$$

3.

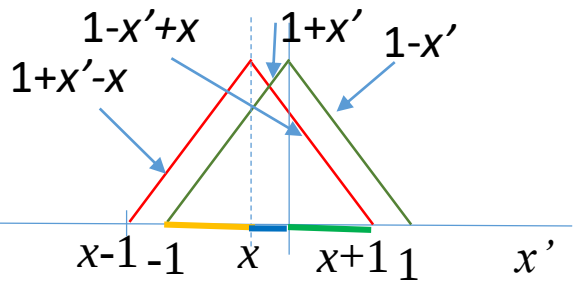


$$0 < x + 1 < 1 \rightarrow -1 < x < 0$$

$$h(x) = \underbrace{\int_{-1}^x (1+x')(1+x'-x) dx'}_{\frac{1}{6}(-x^3 + 3x + 2)} + \underbrace{\int_x^0 (1+x')(1-x'+x) dx'}_{-\frac{1}{6}x(x^2 + 6x + 6)} + \underbrace{\int_0^{x+1} (1-x')(1-x'+x) dx'}_{\frac{1}{6}x(-x^3 + 3x + 2)}$$

$$h(x) = \frac{1}{3}(-x^3 + 3x + 2) - \frac{1}{6}x(x^2 + 6x + 6)$$

3.



$$0 < x + 1 < 1 \rightarrow -1 < x < 0$$

$$h(x) = \underbrace{\int_{-1}^x (1+x')(1+x'-x) dx'}_{\frac{1}{6}(-x^3 + 3x + 2)} + \underbrace{\int_x^0 (1+x')(1-x'+x) dx'}_{-\frac{1}{6}x(x^2 + 6x + 6)} + \underbrace{\int_0^{x+1} (1-x')(1-x'+x) dx'}_{\frac{1}{6}x(-x^3 + 3x + 2)}$$

$$h(x) = \frac{1}{3}(-x^3 + 3x + 2) - \frac{1}{6}x(x^2 + 6x + 6)$$

4. $-1 < x - 1 < 0 \rightarrow 0 < x < 1$

Symetrycznie do 3

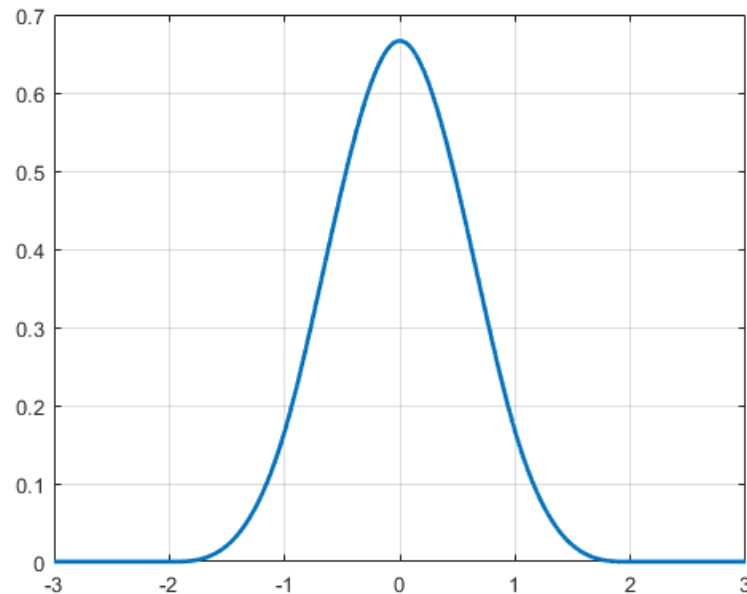
5. $0 < x - 1 < 1 \rightarrow 1 < x < 2$

Symetrycznie do 2

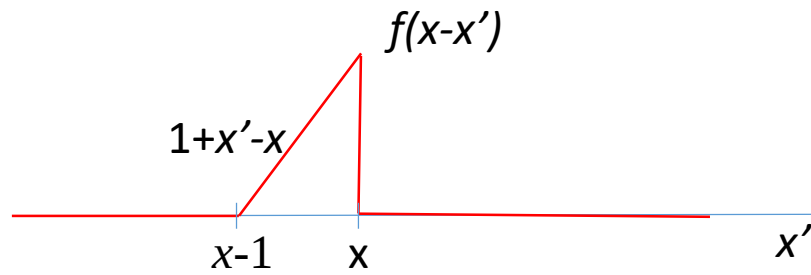
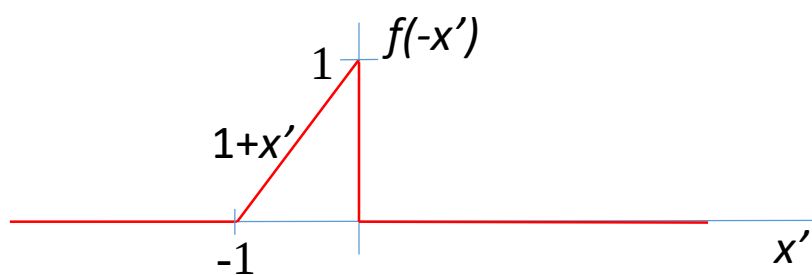
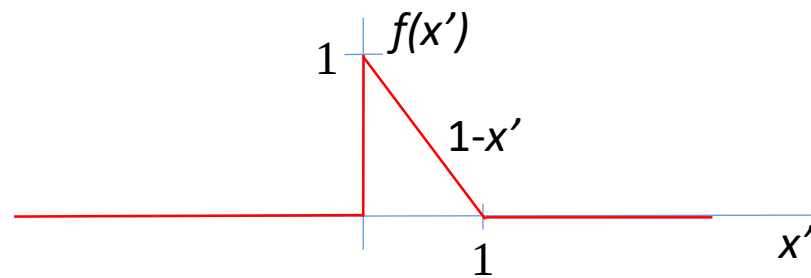
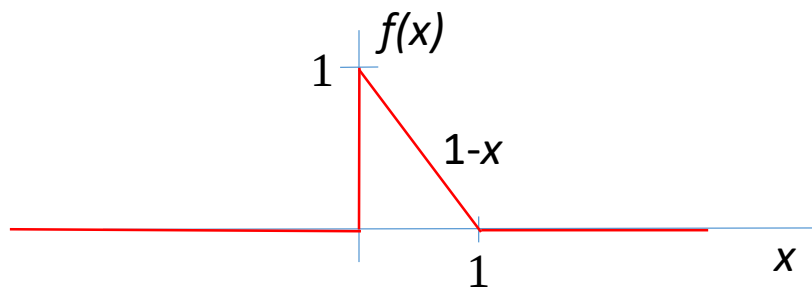
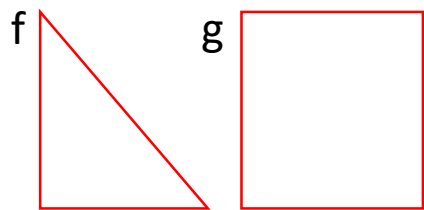
6. $1 < x - 1 \rightarrow 2 < x$

Symetrycznie do 1

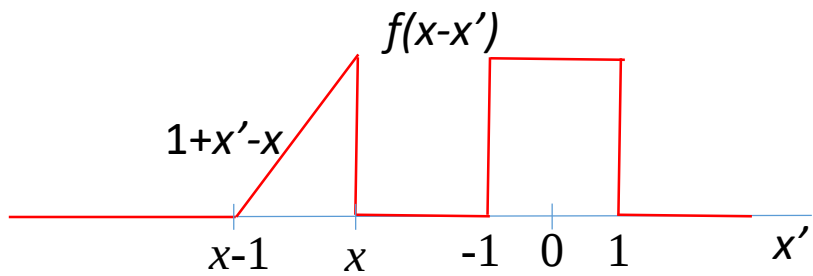
$$h(x) = \begin{cases} 0 & x < -2 \\ \frac{1}{6}(x+2)^3 & -2 < x < -1 \\ \frac{1}{3}(-x^3 + 3x + 2) - \frac{1}{6}x(x^2 + 6x + 6) & -1 < x < 0 \\ \frac{1}{3}(x^3 - 3x + 2) + \frac{1}{6}x(x^2 - 6x + 6) & 0 < x < 1 \\ -\frac{1}{6}(x-2)^3 & 1 < x < 2 \\ 0 & 2 < x \end{cases}$$



Splot



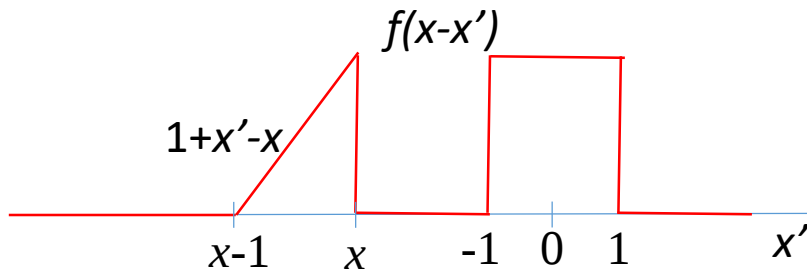
1.



$$x < -1$$

$$h(x) = 0$$

1.



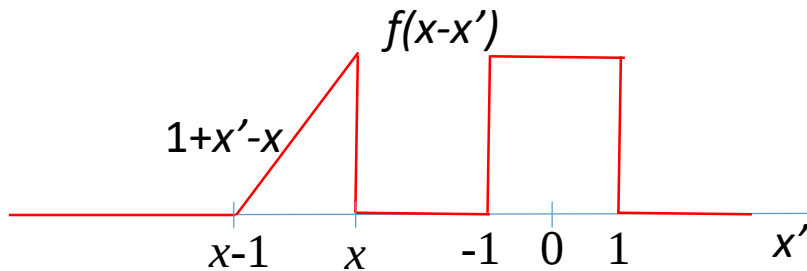
$$x < -1$$

$$h(x) = 0$$

2. $-1 < x < 0$

$$\begin{aligned} h(x) &= \int_{-1}^x (1 + x' - x) dx' = (1 - x) \int_{-1}^x dx' + \int_{-1}^x x' dx' = \\ &= (1 - x)(x + 1) + \frac{x^2}{2} + \frac{1}{2} = \frac{1}{2}(1 - x^2) \end{aligned}$$

1.



$$x < -1$$

$$h(x) = 0$$

2. $-1 < x < 0$

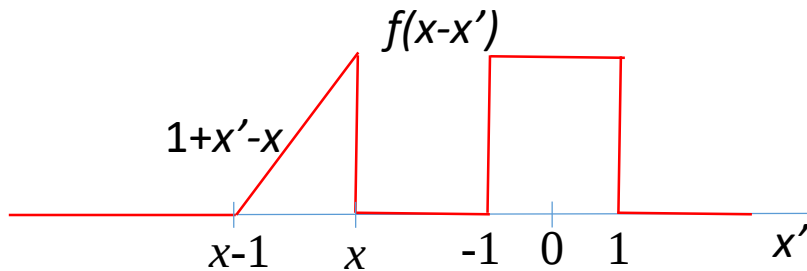
$$h(x) = \int_{-1}^x (1 + x' - x) dx' = (1 - x) \int_{-1}^x dx' + \int_{-1}^x x' dx' =$$

$$= (1 - x)(x + 1) + \frac{x^2}{2} + \frac{1}{2} = \frac{1}{2}(1 - x^2)$$

3. $0 < x < 1$

$$h(x) = \frac{1}{2}$$

1.



$$x < -1$$

$$h(x) = 0$$

2. $-1 < x < 0$

$$\begin{aligned} h(x) &= \int_{-1}^x (1 + x' - x) dx' = (1 - x) \int_{-1}^x dx' + \int_{-1}^x x' dx' = \\ &= (1 - x)(x + 1) + \frac{x^2}{2} + \frac{1}{2} = \frac{1}{2}(1 - x^2) \end{aligned}$$

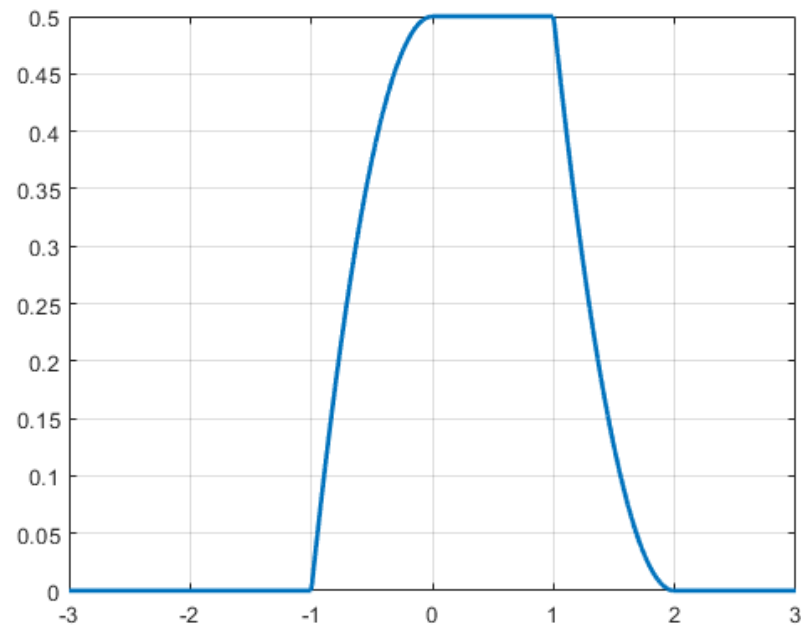
3. $0 < x < 1$

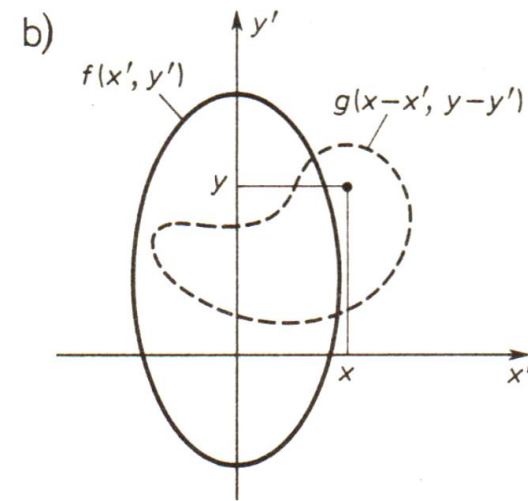
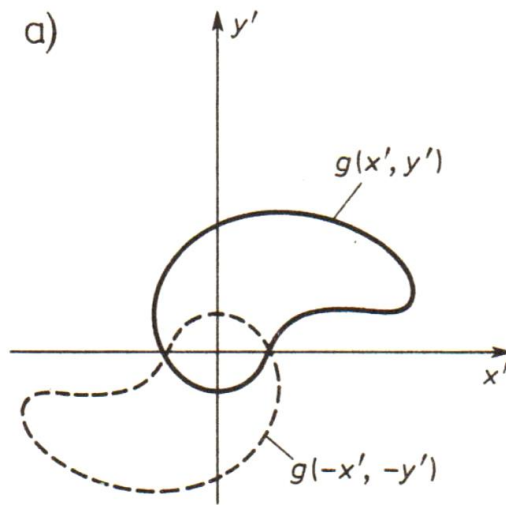
$$h(x) = \frac{1}{2}$$

4. $0 < x - 1 < 1 \rightarrow 1 < x < 2$

$$\begin{aligned} h(x) &= \int_{x-1}^1 (1 + x' - x) dx' = (1 - x) \int_{x-1}^1 dx' + \int_{x-1}^1 x' dx' = \\ &= (1 - x)(2 - x) + \frac{1}{2} [1 - (x - 1)^2] = \frac{x^2}{2} - 2x + 2 \end{aligned}$$

$$h(x) = \begin{cases} 0 & x < -1 \\ \frac{1}{2}(1 - x^2) & -1 < x < 0 \\ \frac{1}{2} & 0 < x < 1 \\ \frac{x^2}{2} - 2x + 2 & 1 < x < 2 \\ 0 & 2 < x \end{cases}$$





$$\varphi(x) = \int_{-\infty}^{\infty} f(x')g(x' - x)dx'$$

$$\varphi(x) = f(x) \star g(x)$$

Splot:

$$h(x) = \int_{-\infty}^{\infty} f(x - x')g(x')dx'$$

$$\varphi(x) = \int_{-\infty}^{\infty} f(x')g(x' - x)dx'$$

$$\varphi(x) = f(x) \star g(x)$$

Splot:

$$h(x) = \int_{-\infty}^{\infty} f(x - x')g(x')dx'$$

Własności:

Korelacja nie jest przemienna:

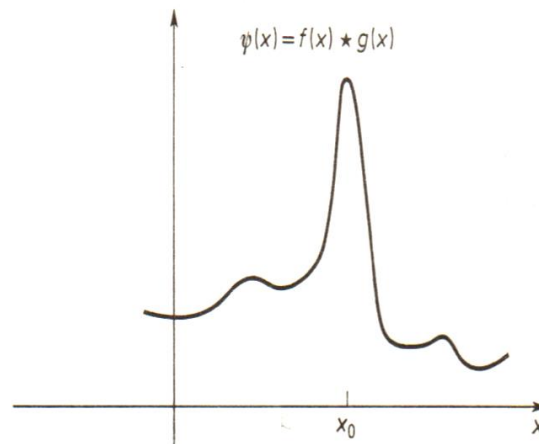
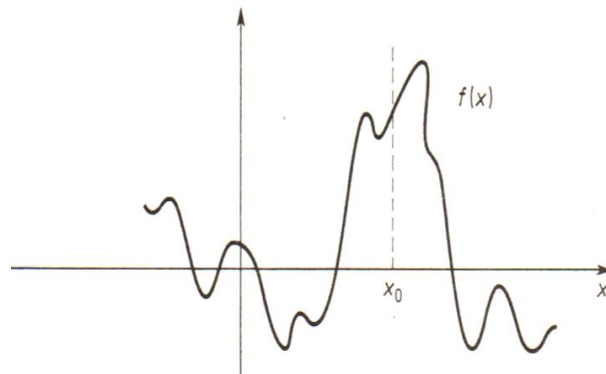
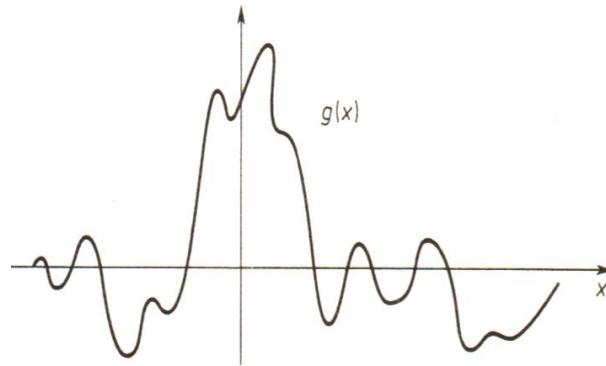
$$f(x) \star g(x) \neq g(x) \star f(x)$$

Korelacja jest równa splotowi z odwróconą funkcją g :

$$f(x) \star g(x) = f(x) \otimes g(-x)$$

Gdy $g(x)$ jest funkcją parzystą to korelacja jest równoważna splotowi

Korelacja



Autokorelacja

Autokorelacja gdy $g(x) = f(x)$

Współczynnik autokorelacji: $\gamma(x) = \frac{\varphi(x)}{\varphi(0)}$

Moduł autokorelacji osiąga największą wartość w ,0' : $|\varphi(x)| \leq \varphi(0)$

Szereg Taylora:

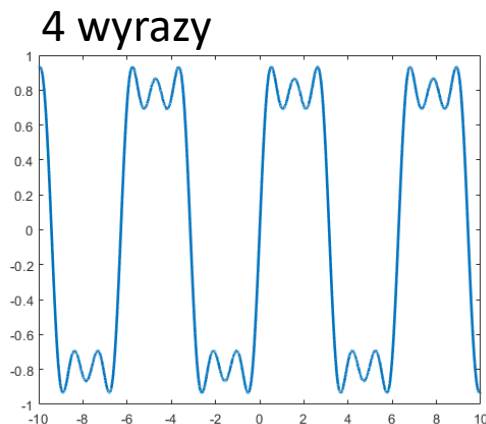
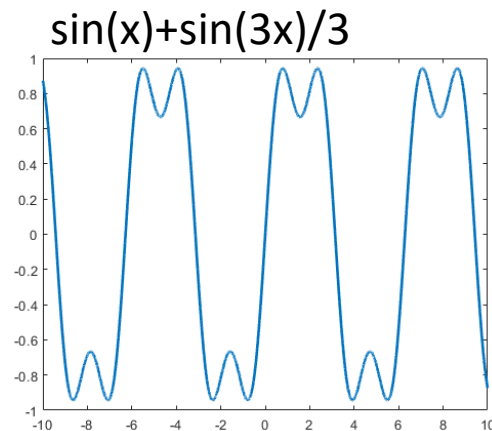
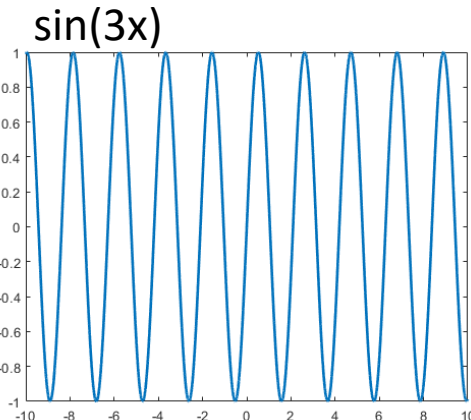
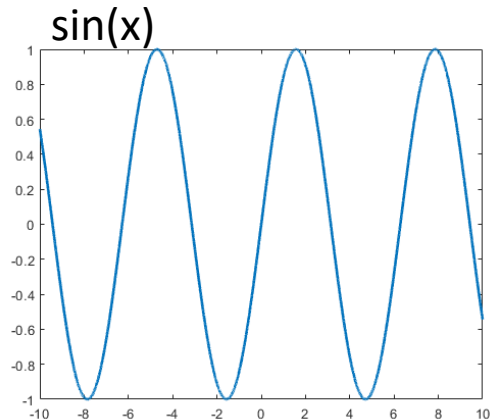
$$\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^{2n-1}}{(2n-1)!}$$

Transformacja Fouriera

Szereg Taylora:

$$\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^{2n-1}}{(2n-1)!}$$

Suma sinusów:



$$\sum_{n=1}^N \frac{\sin[(2n-1)x]}{2n-1}$$

Transformacja Fouriera

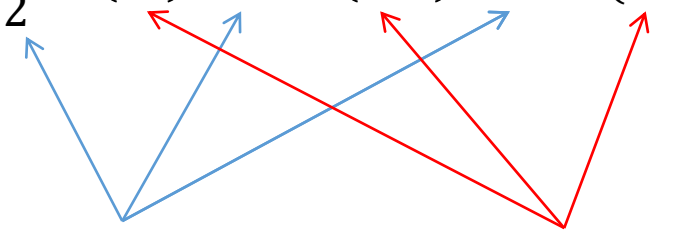
Funkcję periodyczną można przedstawić jako sumę sinusów:

$$f(t) = \sum_{k=1}^n [A_k \sin(2\pi\omega_k t) + B_k \sin(2\pi\omega_k t + \pi/2)]$$

Wygodniej to przepisać jako sumę sinusa i cosinusa:

$$f(t) = \sum_{k=1}^n [A_k \cos(2\pi\omega_k t) + B_k \sin(2\pi\omega_k t)]$$

Np. dla funkcji:

$$f_1 = \frac{1}{2} \sin(\pi t) + 2 \sin(4\pi t) + 4 \cos(2\pi t)$$


amplitudy

częstości

Transformacja Fouriera

Zapis zespolony funkcji trygonometrycznych:

$$e^{i\varphi} = \cos(\varphi) + i \sin(\varphi) \qquad e^{-i\varphi} = \cos(\varphi) - i \sin(\varphi)$$

$$\cos(\varphi) = \frac{e^{i\varphi} + e^{-i\varphi}}{2} \qquad \sin(\varphi) = \frac{e^{i\varphi} - e^{-i\varphi}}{2}$$

Czyli $f(t) = \sum_{k=1}^n [A_k \cos(2\pi\omega_k t) + B_k \sin(2\pi\omega_k t)]$ mogą zapisać jako:

$$f(t) = \sum_{k=1}^n \left[\frac{A_k}{2} (e^{2\pi i \omega_k t} + e^{-2\pi i \omega_k t}) + \frac{B_k}{2} (e^{2\pi i \omega_k t} - e^{-2\pi i \omega_k t}) \right]$$

Podstawiam:

$$C_k = \begin{cases} \frac{A_k - iB_k}{2} & \text{dla } k > 0 \\ \frac{A_k + iB_k}{2} & \text{dla } k < 0 \end{cases} \qquad \omega_k = \omega_{-k} \quad \text{dla } k < 0$$

Transformacja Fouriera

Dostaję:

$$f(t) = \sum_{k=-n}^n [C_k e^{2\pi i \omega_k t}]$$

Czyli nasza funkcja:

$$f_1 = \frac{1}{2} \sin(\pi t) + 2 \sin(4\pi t) + 4 \cos(2\pi t)$$

k	Częstotliwość (ω_k)	C_k
3	2	1
2	1	2
1	1/2	1/4
0	0	0
-1	-1/2	-1/4
-2	-1	2
-3	-2	-1

Transformacja Fouriera

$$f_1 = \boxed{\frac{1}{2} \sin(\pi t)} + 2 \sin(4\pi t) + 4 \cos(2\pi t)$$

$$C_{-k} \exp[2\pi i \omega t] + C_k \exp[-2\pi i \omega t]$$

$$C_{-1} \exp[2\pi i \omega t] + C_1 \exp[-2\pi i \omega t] =$$

$$C \exp[2\pi i \omega t] - C \exp[-2\pi i \omega t] =$$

$$\sin(\varphi) = \frac{e^{i\varphi} - e^{-i\varphi}}{2}$$

$$\boxed{2C} \left[\frac{\exp[2\pi i \omega t] - \exp[-2\pi i \omega t]}{2} \right]$$

$$\begin{array}{l} \downarrow \\ \frac{1}{2} \\ \downarrow \\ C = \frac{1}{4} \end{array} \quad \begin{array}{l} \downarrow \\ \sin(\pi t) \\ \downarrow \\ \omega = \frac{1}{2} \end{array}$$

k	Częstotliwość (ω_k)	C_k
3	2	1
2	1	2
1	1/2	1/4
0	0	0
-1	-1/2	-1/4
-2	-1	2
-3	-2	-1

Transformacja Fouriera

$$f_1 = \frac{1}{2} \sin(\pi t) + 2 \sin(4\pi t) + 4 \cos(2\pi t)$$

$$C_{-k} \exp[2\pi i \omega t] + C_k \exp[-2\pi i \omega t]$$

$$C_{-2} \exp[2\pi i \omega t] + C_2 \exp[-2\pi i \omega t] =$$

$$C \exp[2\pi i \omega t] + C \exp[-2\pi i \omega t] =$$

$$\cos(\varphi) = \frac{e^{i\varphi} + e^{-i\varphi}}{2}$$

$$2C \left[\frac{\exp[2\pi i \omega t] + \exp[-2\pi i \omega t]}{2} \right]$$

Diagram illustrating the derivation of C and ω from the boxed expression:

- The coefficient $2C$ is boxed in red. An arrow points from it to the value 4 , which then points to $C = 2$.
- The expression $\frac{\exp[2\pi i \omega t] + \exp[-2\pi i \omega t]}{2}$ is boxed in green. An arrow points from it to $\cos(2\pi t)$, which then points to $\omega = 1$.

k	Częstotliwość (ω_k)	C_k
3	2	1
2	1	2
1	1/2	1/4
0	0	0
-1	-1/2	-1/4
-2	-1	2
-3	-2	-1

Transformacja Fouriera

$$f_1 = \frac{1}{2} \sin(\pi t) + 2 \sin(4\pi t) + 4 \cos(2\pi t)$$

$$C_{-k} \exp[2\pi i \omega t] + C_k \exp[-2\pi i \omega t]$$

$$C_{-3} \exp[2\pi i \omega t] + C_3 \exp[-2\pi i \omega t] =$$

$$C \exp[2\pi i \omega t] - C \exp[-2\pi i \omega t] =$$

$$\sin(\varphi) = \frac{e^{i\varphi} - e^{-i\varphi}}{2}$$

$$2C \left[\frac{\exp[2\pi i \omega t] - \exp[-2\pi i \omega t]}{2} \right]$$

Diagram illustrating the identification of parameters from the Fourier series expression:

- The coefficient $2C$ is identified as 2 , leading to $C = 1$.
- The argument $\sin(4\pi t)$ is identified as $\sin(\omega t)$, leading to $\omega = 2$.

k	Częstotliwość (ω_k)	C_k
3	2	1
2	1	2
1	1/2	1/4
0	0	0
-1	-1/2	-1/4
-2	-1	2
-3	-2	-1

Transformacja Fouriera

Przechodzimy do funkcji ciągłej:

$$f(t) = \int_{-\infty}^{\infty} F(\omega) e^{2\pi i \omega t} d\omega$$

ODWROTNA TRANSFORMATA FOURIERA

Składowe częstotliwości $\sim C_k$

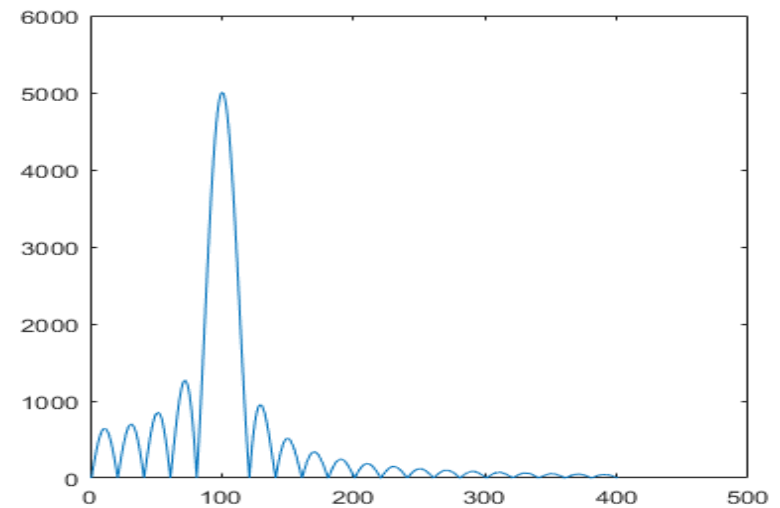
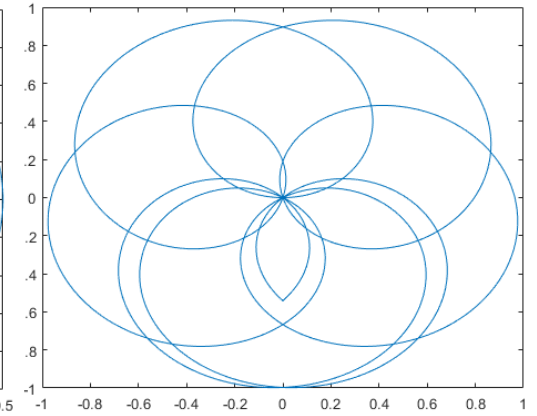
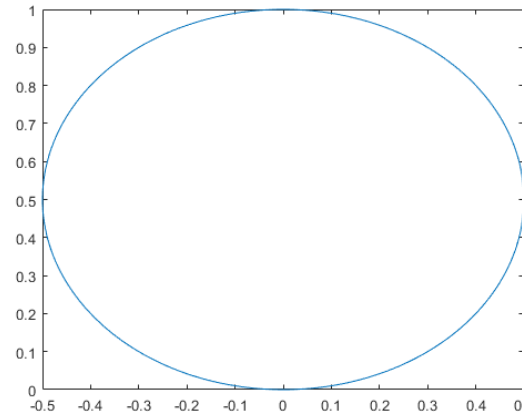
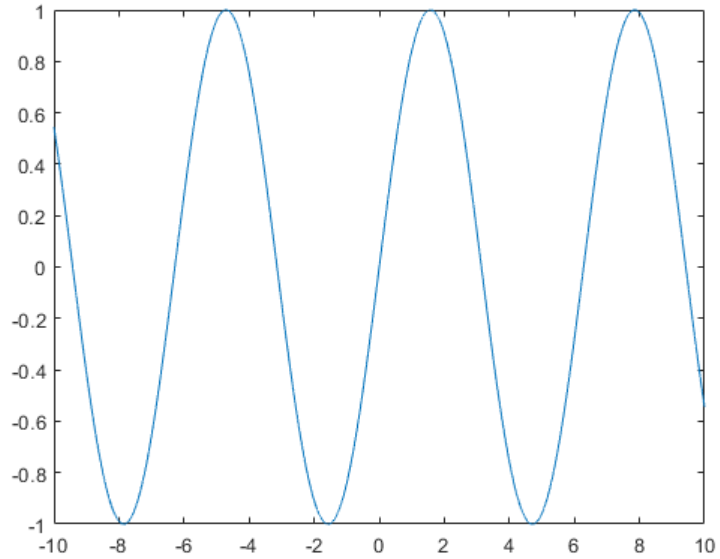


$$F(t) = \int_{-\infty}^{\infty} f(\omega) e^{-2\pi i \omega t} d\omega$$

TRANSFORMATA FOURIERA

Transformacja Fouriera

Analogia z centrum masy



Transformacja Fouriera 2D

$$F(v_x, v_y) = \iint_{-\infty}^{\infty} f(x, y) \exp[-i2\pi(xv_x + yv_y)] dx dy$$

transformata

$$f(x, y) = \iint_{-\infty}^{\infty} F(v_x, v_y) \exp[i2\pi(xv_x + yv_y)] dv_x dv_y$$

transformata odwrotna

$$f(x, y) \Leftrightarrow F(v_x, v_y)$$

Transformacja Fouriera 2D

$$f(x, y) \Leftrightarrow F(v_x, v_y) \quad g(x, y) \Leftrightarrow G(v_x, v_y)$$

Właściwości:

Twierdzenie o liniowości:

$$\mathcal{F}\{af(x, y) + bg(x, y)\} = aF(v_x, v_y) + bG(v_x, v_y)$$

Twierdzenie o podobieństwie (skali):

$$\mathcal{F}\{f(ax, by)\} = \frac{1}{|ab|} F\left(\frac{v_x}{a}, \frac{v_y}{b}\right)$$

Twierdzenie o przesunięciu w przestrzeni położeń:

$$\mathcal{F}\{f(x - x_0, y - y_0)\} = F(v_x, v_y) \exp[-i2\pi(x_0 v_x + y_0 v_y)]$$

Twierdzenie o przesunięciu w przestrzeni częstości:

$$\mathcal{F}\{f(x, y) \exp[i2\pi(xv_1 + yv_2)]\} = F(v_x - v_1, v_y - v_2)$$

Twierdzenie o wzajemności transformat:

$$\mathcal{F}\mathcal{F}\{f(x, y)\} = f(-x, -y) \quad \mathcal{F}^{-1}\mathcal{F}\{f(x, y)\} = f(x, y)$$

Transformacja Fouriera 2D

Twierdzenie o splocie w przestrzeni położeń:

$$\mathcal{F}\{f(x, y) \otimes g(x, y)\} = F(v_x, v_y)G(v_x, v_y)$$

$$f(x, y) \star g(x, y) = \mathcal{F}^{-1}\{F(v_x, v_y)G(v_x, v_y)\}$$

Twierdzenie o splocie w przestrzeni częstości:

$$\mathcal{F}\{f(x, y)g(x, y)\} = F(v_x, v_y) \star G(v_x, v_y)$$

Twierdzenie o korelacji wzajemnej:

$$\mathcal{F}\{f(x, y) \star g^*(x, y)\} = F(v_x, v_y)G^*(v_x, v_y)$$

$$\mathcal{F}\{f(x, y) \star f^*(x, y)\} = |F(v_x, v_y)|^2$$

Twierdzenie o mocy:

$$\iint_{-\infty}^{\infty} |f(x, y)|^2 dx dy = \iint_{-\infty}^{\infty} |F(v_x, v_y)|^2 dv_x dv_y$$

Transformacja Fouriera 2D

Pary transformat:

$$f(x, y) \Leftrightarrow F(v_x, v_y)$$

$$\text{sgn}(x, y) = \text{sgn}(x)\text{sgn}(y) \Leftrightarrow \frac{1}{i\pi v_x} \frac{1}{i\pi v_y}$$

$$\text{rect}(x, y) = \text{rect}(x)\text{rect}(y) \Leftrightarrow \text{sinc}(v_x)\text{sinc}(v_y) = \text{sinc}(v_x, v_y)$$

$$\Lambda(x, y) = \Lambda(x)\Lambda(y) \Leftrightarrow \text{sinc}^2(v_x)\text{sinc}^2(v_y) = \text{sinc}^2(v_x, v_y)$$

$$\delta(x, y) \Leftrightarrow 1$$

$$\text{comb}(x, y) = \text{comb}(x)\text{comb}(y) \Leftrightarrow \text{comb}(v_x)\text{comb}(v_y) = \text{comb}(v_x, v_y)$$

$$\exp[i\pi(x + y)] \Leftrightarrow \delta\left(v_x - \frac{1}{2}, v_y - \frac{1}{2}\right)$$

$$\exp[-\pi(x^2 + y^2)] \Leftrightarrow \exp[-\pi(v_x^2 + v_y^2)]$$

$$\text{circ}(r) = \text{circ}(\sqrt{x^2 + y^2}) \Leftrightarrow \frac{J_1(2\pi\rho)}{\rho} = \frac{J_1(2\pi\sqrt{v_x^2 + v_y^2})}{\sqrt{v_x^2 + v_y^2}}$$