Mixing and diffusion in planar vortices

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Abstract This abstract is 200 words long. If a passive scalar field such as dye or temperature is placed in a smooth planar vortex, for example a Gaussian monopole, the scalar becomes wound up into a spiral structure because of differential rotation. An analogous process occurs if weak non-axisymmetric vorticity is introduced, for example by perturbing the vortex using an external irrotational flow. Although the wind-up of vorticity and scalar look very similar, there are a number of differences because the vorticity is coupled back to the flow field, and this is important close to the centre of the vortex. We show that this leads to rapid suppression of vorticity and some surprising power laws. We also discuss the way in which differential rotation enhances dissipation of scalar and vorticity, and the relevant time-scales, especially close to the centre of the vortex.

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Here's a blob that's subjected to swirl;
It's a problem for somebody virile!
But right at the core
Where it turns more and more,
That's where I get in a whirl.

1. Introduction

The interaction of fluid motion and molecular diffusion can accelerate the destruction of advected quantities, such as passive scalars, magnetic fields and vorticity itself. One particularly important example is when the flow is in the plane and possesses a region of closed stream lines, for example a smooth flow having circular stream lines. In this case differential rotation in the flow tends to reduce radial scales, and to enhance diffusion in the radial direction, which now occurs on a new time-scale intermediate between the turn-over time-scale t and the long time-scale of molecular diffusion T. This new accelerated diffusion time-scale is of order $T^{1/3}t^{2/3}$.

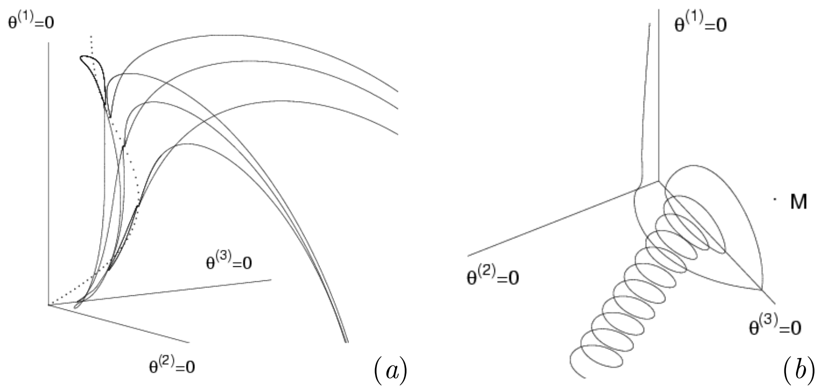

Figure 1. Streamline patterns for different kinds of motion of the wall: (a) translating wall; (b) rotating wall.

Let us introduce the usual dimensionless numbers for the ratio T/t, namely, a Péclet number Pe for a passive scalar, a Reynolds number Re for vorticity and a magnetic Reynolds number Rm for magnetic fields. Taking the turn-over time t to be unity without loss of generality, the time-scales for accelerated diffusion become

$$\mathrm{Pe}^{1/3}, \qquad \mathrm{Re}^{1/3}, \qquad \mathrm{Rm}^{1/3}, \tag{1}$$

respectively.

Table 1. Small Table

one	*two*	*three*
C	D	E

These rapid time-scales were explained analytically in a clutch of papers in the early eighties (Lundgren 1982, Moffatt & Kamkar 1983, Rhines & Young 1983). They were first observed numerically rather earlier by Weiss (1966), who considered magnetic field evolution in the kinematic approximation at high Rm. In this case the vector potential for the magnetic field is advected as a passive scalar and the destruction of this scalar on the $\mathrm{Rm}^{1/3}$ time-scale leads to the process of flux expulsion of the magnetic field. Bernoff & Lingevitch (1994) observed the analogous $\mathrm{Re}^{1/3}$ time-scale for vorticity in numerical simulations of the relaxation of perturbations to a Gaussian vortex. They also gave the term 'shear–diffuse mechanism' to this accelerated diffusion process.

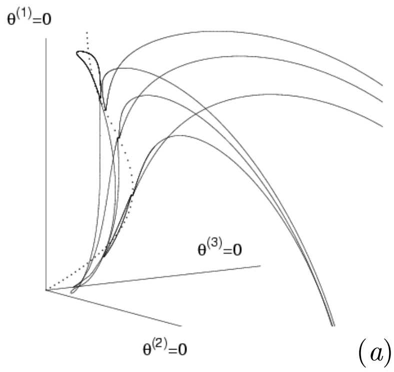

Figure 2a. Streamline patterns for translating wall.

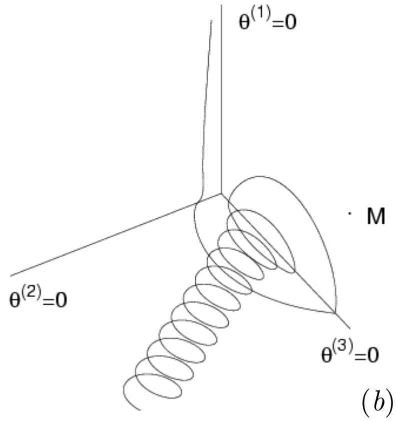

Figure 2b. Streamline patterns for rotating wall.

The rapid time-scales arise from the interaction of diffusion and differential rotation, and so the magnitude of the differential rotation enters into estimates of the time-scale. If $\alpha(r)$ is the angular velocity of fluid elements around the circular stream lines, then a stricter estimate of the time-scale for the destruction of scalar by the shear–diffuse mechanism at a given radius r is

$$\alpha'(r)^{-2/3}\,\mathrm{Pe}^{1/3}, \tag{2}$$

(and analogously for magnetic field or vorticity). This dependence on the magnitude of the differential rotation has been considered by Flohr & Vassilicos (1997) who note that if $\alpha'(r)$ has a wide variation with r then the above estimate gives a wide range of time-scales and introduces new scaling laws for diffusion. An example of this would be when the flow field is a point vortex, for which $\alpha'(r)$ diverges as r tends to zero (Bajer 1998).

As noted by many authors the shear–diffuse mechanism requires $\alpha'(r)$ to be non-zero, and correspondingly the time-scale above in () diverges if $\alpha'(r) = 0$ at a given radius. In this case the interaction of shear and diffusion is less potent, and longer time-scales are introduced. For example Parker (1966) studies flux expulsion in an extreme case in which the flow field is solid body rotation within a cylinder, the fluid outside being at rest; the time-scale of diffusion becomes of order Rm.

Table 2. Effects of the Two Types of Scaling Proposed by Dennard and Co-Workers.[a,b]

Parameter	κ *Scaling*	κ, λ *Scaling*
Dimension	κ^{-1}	λ^{-1}
Voltage	κ^{-1}	κ^{-1}
Currant	κ^{-1}	λ/κ^2
Dopant Concentration	κ	λ^2/κ

[a]Refs. 19 and 20.
[b]$\kappa, \lambda > 1$.

Now in this paper we consider only smooth flow fields, and these have the property that $\alpha'(r)$ is necessarily zero at the origin $r = 0$. In fact in any region of closed stream lines of a smooth flow the shear–diffuse time-scale () is non-uniform at the origin, and diffusion of a scalar, magnetic field or vorticity must occur on a longer time-scale. The aim of the present paper is to obtain this new, longer time-scale and to determine the ultimate fate of a passive scalar, magnetic field or vorticity near the origin, for this case of smooth fluid flow.

References

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