Energy, helicity and crossing number relations for complex flows

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Abstract Algebraic and topological measures based on crossing number relations provide bounds on energy and helicity of ideal fluid flows and can be used to quantify morphological complexity of tangles of magnetic and vortex tubes. In the case of volume-preserving flows we discuss new results useful to determine lower bounds on magnetic energy in terms of topological crossing number and average spacing of the physical system. New relationships between average crossing number, energy and helicity are derived also for homogeneous vortex tangles. These results find interesting applications in the study of possible connections between energy and complexity of structured flows.

Topological arguments show
That the energy's bounded below;
But what's so engrossing's
The number of crossings,
From which my new insights will flow.

1. Magnetic and vortex knots as standard embeddings

Consider an incompressible and perfectly conducting fluid in an unbounded domain $\mathcal{D}$ of $\mathbb{R}^3$ that is simply connected, with fluid velocity $\boldsymbol{u} = \boldsymbol{u}(\mathbf{x}, t)$, smooth function of the position vector $\mathbf{x}$ and time t and such that $\nabla \cdot \boldsymbol{u} = 0$ in $\mathcal{D}$ and $\boldsymbol{u} = 0$ at infinity. Consider the class of magnetic fields $\{\mathbf{B}\}$ that are solenoidal and frozen in $\mathcal{D}$, that is

$$\mathbf{B} \in \{\nabla \cdot \mathbf{B} = 0 \quad \text{and} \quad \partial_t \mathbf{B} = \nabla \times (\boldsymbol{u} \times \mathbf{B})\} \, . \tag{1}$$

Under ideal conditions these fields have their prescribed topology conserved during time evolution. We restrict our attention to fields that are localised in space and that are indeed confined to tubular neighbourhoods of knots and links. By construction the field is standardly embedded into nested tori T_i centred on smooth loops $\mathcal{C}_i$ that are knot-