Helicity conservation laws

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Abstract Using the language of differential forms on a space-time, one can write the equation of an ideal fluid in a form similar to the Maxwell equations. Vorticity current plays then the role of the source term and the Euler equations can be interpreted as the generalisation, to the whole space-time, of the well-known fact that the number of vortex lines passing through any two-dimensional surface spanned on a closed contour can be expressed by a circulation associated with this contour. A similar procedure can be used for the ideal MHD. It appears that by using this formulation various helicity conservation theorems may be derived in the natural and straightforward manner.

My passion for diff'rential forms
Defies all traditional norms;
It may make you queasy,
But it's really quite easy;
It ought to be taught in the dorms.

1. Introduction

It appears that many equations of mathematical physics when formulated in the language of differential forms take especially simple and elegant form (Flanders 1963, Arnold, Khesin 1998). This, as we will see, concerns also equations of fluid dynamics (Peradzyński 1990,1991), including MHD. The simplest example of a differential form (apart from 0-forms which are just functions) is a 1-form $\alpha = \alpha_\nu \, dx^\nu$, i.e. a field of covectors. Such a form can be evaluated on a vector-field X to obtain $\alpha(X) = \alpha_\nu X^\nu$ (with Einstein summation convention observed). Having k such forms $\alpha^1, \ldots, \alpha^k$ one defines their exterior product $\alpha^1 \wedge \ldots \wedge \alpha^k$ – a simple k-form which when evaluated on vector-fields $X_1, \ldots, X_k$ gives

$$(\alpha^1 \wedge \ldots \wedge \alpha^k)\,(X_1, \ldots, X_k) = \frac{1}{k!} \det \; [\alpha^i(X_j)].$$