Diffusion of Lagrangian invariants in the Navier-Stokes equations

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Abstract The incompressible Euler equations can be written as the active vector system

$$(\partial_t + u \cdot \nabla) A = 0$$

where $u = W[A]$ is given by the Weber formula

$$W[A] = \mathbf{P} \{(\nabla A)^* v\}$$

in terms of the gradient of A and the passive field $v = u_0(A)$. ($\mathbf{P}$ is the projector on the divergence-free part.) The initial data is $A(x, 0) = x$, so for short times this is a distortion of the identity map. After a short time one obtains a new u and starts again from the identity map, using the new u instead of u_0 in the Weber formula. The viscous Navier-Stokes equations admit the same representation, with a diffusive back-to-labels map A and a v that is no longer passive.

I'll analyse Navier-Stokes,
For dynamics of vodkas and cokes;
  [A] *is an entity*
  Near the identity;
That's how I'll baffle you folks!

1. Introduction

The classical method of characteristics for partial differential equations (John 1991) allows one to prove the existence of finite time singularities for hyperbolic conservation laws. The classical maximum principle allows one to prove that singularities are absent in certain nonlinear parabolic equations. The equations of incompressible fluids do not fit neatly in either of these classes of PDE. Ideal, frictionless fluids in Eulerian coordinates are a hyperbolic system with non-local additive forcing due to the gradient of pressure. The gradients of velocity are affected by non-local fluctuations in the pressure Hessian, and the classical blow-up proofs do not apply. Viscous, incompressible fluids in Eulerian coor-