An example of development of singularity in a solution to the force-free Euler equation

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A mixed Euler-Lagrangian description of vortex line transport by a force-free non-viscous fluid flow was recently proposed by Kuznetsov & Ruban [1]. The motion of Lagrangian markers $\mathbf{R}(\mathbf{a},t)$ satisfies

$$\frac{\partial \mathbf{R}}{\partial t} = \mathbf{v}(\mathbf{R},t) - \frac{\mathbf{\Omega}(\mathbf{R},\mathbf{t})\cdot\mathbf{v}(\mathbf{R},\mathbf{t})}{|\mathbf{\Omega}(\mathbf{R},\mathbf{t})|^{\mathbf{2}}}\mathbf{\Omega}(\mathbf{R},t). \tag{1}$$

Here $\mathbf{a}$ is the initial position of a marker: $\mathbf{R}(\mathbf{a},0)=\mathbf{a}$, t is time, $\mathbf{v}$ is the flow velocity, and

$$\mathbf{\Omega}(\mathbf{R},t) \equiv \mathrm{curl}_{\mathbf{R}}\mathbf{v}(\mathbf{R},t) \tag{2}$$

is vorticity. Equations (1) and (2) are closed by the relation

$$\mathbf{\Omega}(\mathbf{R}(\mathbf{a},t),t) = \left(\det \parallel \frac{\partial \mathbf{R}}{\partial \mathbf{a}} \parallel\right)^{-1} (\mathbf{\Omega}_0\cdot\nabla_{\mathbf{a}})\mathbf{R}(\mathbf{a},t), \tag{3}$$

where $\mathbf{\Omega}(\mathbf{a},0)=\mathbf{\Omega}_0(\mathbf{a})$ is the initial vorticity.

We have developed a code for numerical solution of the system (1)-(3). 2π-periodicity in space is assumed, so that Fast Fourier Transforms can be used for the inversion of curl in (2). A second-order Adams-Bashforth finite-difference scheme is used for numeric integration of (1). To test the code, we have checked that an ABC flow, which is a steady solution to the force-free Euler equation, in computations remains unaltered. We have also verified that the numerical error in conservation of the total kinetic energy of the flow is $O(h^2)$ (here h is the size of the $\mathbf{a}$-mesh, where $\mathbf{\Omega}_0$ is set); this is consistent with the order of methods applied for spatial discretization of the problem (2) and with finite-difference schemes used to evaluate gradients in (3).

Computations have been performed with a uniform mesh of 128^3 points. Collapse is found to develop in a flow, which is initially chosen to have random Fourier harmonics and an exponentially decaying energy spectrum. The flow does not possess any symmetry. According to the theory of Kuznetsov & Ruban [2], in the region of collapse vorticity behaves as $(t_0 - t)^{-1}$, which is indeed found in computations (see Fig.1). Their another prediction is confirmed: eigenvalues of the Hessian $\| \frac{\partial^2}{\partial \mathbf{a}_i \partial \mathbf{a}_j} |\mathbf{\Omega}|^{-1} \|$ at the point of singularity are non-singular and slightly depend on time.

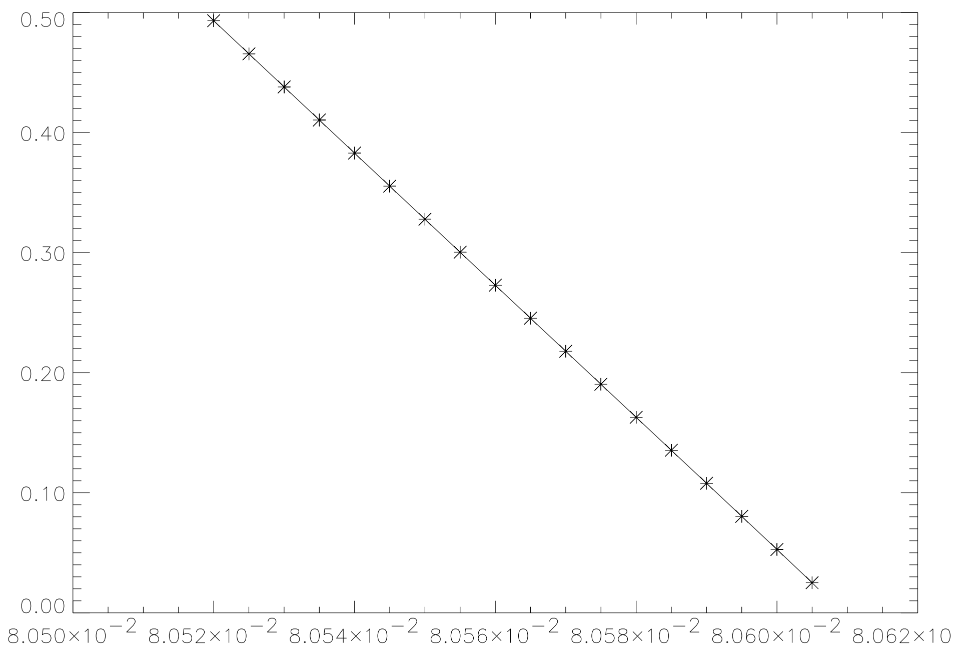

Figure 1: $|\mathbf{\Omega}|^{-1}$ (vertical axis) as a function of time (horizontal axis).

References

[1] Kuznetsov E.A. & Ruban V.P. (2000). Hamiltonian dynamics of vortex and magnetic lines in hydrodynamic type systems. *Phys. Rev.* E, **61**, 831–841.

[2] Kuznetsov E.A. & Ruban V.P. (2001). Breaking of vortex lines – a new mechanism of collapse in hydrodynamics. Submitted.