Motion of vortex lines in the hydrodynamic formulation of quantum mechanics

Cezary Śliwa

Centre for Theoretical Physics, Polish Academy of Sciences
sliwa@cft.edu.pl

In quantum theory, vortex lines appear in the hydrodynamic interpretation of the wave equation. In this interpretation, which is originally due to Madelung [1], the flow of the probability density for a single particle is described in terms of the hydrodynamic variables. In [2], for the sake of simplicity, the standard time-dependent Schrödinger equation, and the related vortex lines embedded in the probability fluid of the quantum particle, were considered. A vortex line in this case is simply the curve defined by equating the wave function to zero. The linearity of the Schrödinger equation enables us to obtain a large family of exact time-dependent analytic solutions for the wave functions with vortex lines. Moreover, the method is general enough to allow for various initial configurations of the vortex lines.

Although the equation of motion of the quantum mechanical probability fluid is different in its literal form from the equations describing the real physical fluid, we believe that the evolution of the vorticity in the quantum and in the real fluid share the same qualitative features that can be described in terms of the topology of the vortex lines configurations. In particular, the topology of vortex motion should remain unchanged even if a more complicated quantum wave equations were considered, such as the relativistic Klein–Gordon equations. The general phenomena such as the switchover, creation and annihilation of vortices can be observed in the quantum mechanical fluid.

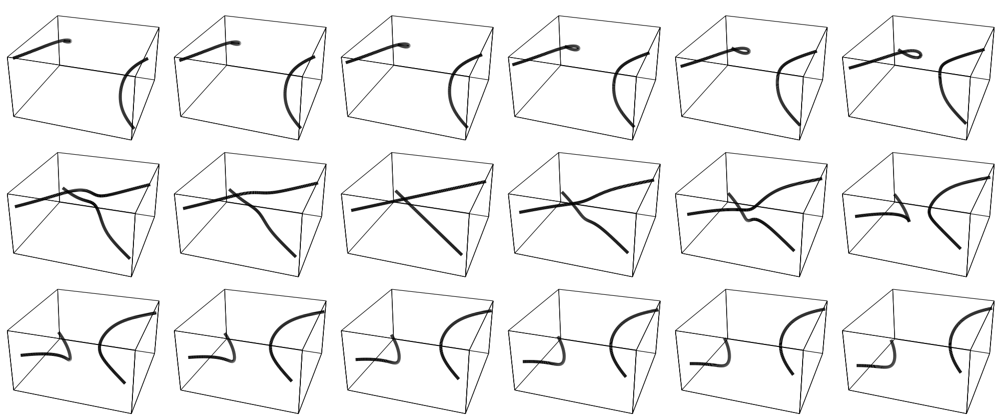

Figure 1: A switchover of two vortex lines embedded in a solution of the free Schrödinger equation. At $t = 0$ there are two non-intersecting straight vortex lines.

On the other hand, the quantum fluid enjoys some peculiar features. One of them is the quantization of vortex strength: the circulation of the velocity field along any closed contour not intersecting the vortex lines is a multiple of $2\pi\hbar/m$. Also, the dynamics of a vortex line depends not only on its shape, but also on the exact expression for the wave function. Suprisingly, the force acting on the quantum particle influences the motion of vortex lines only in an indirect way; the potential

$V(\mathbf{r})$ does not appear in the formula for the velocity $\mathbf{u}$ of a point on the vortex line, since the wave function vanishes on the vortex line.

In [2], a general method was presented for generating solutions to the quantum wave equations. The idea is to differentiate a formula given for a whole family of time-dependent solutions (the generating family) with respect to the continuous parameters of the family, obtaining solutions to the same wave equation again. In the simplest case the generating family is the family of plane waves parameterized by the wave vector $\mathbf{k}$, and differentiating with respect to k_x, k_y or k_z yields a factor of x, y or z. Therefore, any polynomial expression in the coordinates x, y, z can be generated for the initial wave function in this way. The resulting expression defines the vortex lines as algebraic curves in the Euclidean space.

References

[1] Madelung, O. (1926). *Z. Phys* **40**, 342.

[2] Białynicki-Birula, I., Białynicka-Birula, Z. & Śliwa, C. (2000). Motion of vortex lines in quantum mechanics. *Phys. Rev. A* **61**, 032110.