# Finite time singularities in a class of hydrodynamic models

Victor P. Ruban

*Landau Institute for Theoretical Physics, Moscow, Russia*
`ruban@itp.ac.ru`

Dmitry I. Podolsky

`podolsky@itp.ac.ru`

Jens Juul Rasmussen

*Risø National Laboratory, Optics and Fluid Dynamics Department, Denmark*
`jens.juul.rasmussen@risoe.dk`

In this paper [1], we take the point of view that infinite curvature of frozen-in vortex lines is in some sense a more fundamental characteristics of hydrodynamic singularity than infinite value of the vorticity maximum. To illustrate this statement, we

consider a class of models of an incompressible inviscid fluid, different from Eulerian hydrodynamics, such that finite energy solutions with infinitely thin frozen-in vortex filaments of finite strengths are possible. Thus, we deal with a situation when the vorticity maximum is infinite from the very beginning, but nevertheless, this fact itself does not imply a singular behaviour in the dynamics of vortex strings, while their shape is

smooth and the distance between them is finite. However, the interaction between filaments may result in formation of a finite time singularity for the curvature of vortex strings. It is the main purpose of present work to study this phenomenon analytically. In general, our approach is based on the Hamiltonian formalism for frozen-in vortex lines [2]-[4].

It is a well known fact that absence of solutions with singular vortex filaments in Eulerian hydrodynamics is manifested, in particular, as a logarithmic divergency of the corresponding formal expression for the energy functional of an infinitely thin vortex filament having a finite circulation $\Gamma$ and a shape $\mathbf{R}(\xi)$ (this is actually the Hamiltonian functional determining entirely the dynamics of the system [2]-[4]):

$$\mathcal{H}^{\Gamma}\{\mathbf{R}(\xi)\} = \frac{\Gamma^2}{8\pi} \oint\oint \frac{(\mathbf{R}'(\xi_1)\cdot\mathbf{R}'(\xi_2))d\xi_1\, d\xi_2}{|\mathbf{R}(\xi_1)-\mathbf{R}(\xi_2)|} \to \infty. \tag{1}$$

More important is that the self-induced velocity of a curved string in Eulerian hydrodynamics is also infinite. This is the reason, why we cannot work in the framework of Eulerian hydrodynamics with such one-dimensional objects, that are very attractive for theoretical treatment. The situation becomes more favourable, when we consider a class of regularized models, with the divergency of the energy functional eliminated. It should be stressed here that in regularized systems the usual relation $\mathbf{\Omega} = \mathrm{curl}\, \mathbf{v}$ between the vorticity and velocity fields is no more valid, and in this case $\Gamma$ is not the circulation of the velocity around the filament, but it is the circulation of the canonical momentum field (see the cited papers for more details). However, dynamical properties of a de-singularized system depend on the manner of regularization. For instance, it is possible to replace the singular Green's function $G(|\mathbf{R}_1 - \mathbf{R}_2|)$ in (1) (where $G(r) \sim 1/r$) by some analytical function which has no singular points near the real axis in the complex plane (for examples by $G_q(r) \sim \tanh(qr)/r$ or by $G_\epsilon(r) \sim 1/\sqrt{r^2 + \epsilon^2}$). In that case we may not expect any finite time singularity formation, because the corresponding velocity field created by the vortex string appears to be too smooth with any shape of the curve, and this fact prevents drawing together some pieces of the string. With such a very smooth velocity field, a singularity formation needs an infinite time.

In this paper we consider another type of regularization of the Hamiltonian functional, when the Green's function is still singular, but this singularity is integrable in the contour integral analogous to the expression (1):

$$\mathcal{H}_\alpha^\Gamma\{\mathbf{R}(\xi)\} \sim \frac{\Gamma^2}{2} \oint\oint \frac{(\mathbf{R}'(\xi_1) \cdot \mathbf{R}'(\xi_2)) d\xi_1 \, d\xi_2}{|\mathbf{R}(\xi_1) - \mathbf{R}(\xi_2)|^{1-\alpha}}, \tag{2}$$

with a small but finite positive constant $0 < \alpha \ll 1$. If $\alpha$ is not small, we actually have models that are rather different from Eulerian hydrodynamics. Nevertheless, such models

still have many common features with usual hydrodynamics, which are important for singularity formation in the process of the interaction between vortex filaments: a similar hydrodynamic type structure of the Hamiltonian and a power-like behaviour of the Green's function, with negative exponent. Therefore we believe that it is useful to investigate these models, especially the question about the formation of a finite time singularity in

the vortex line curvature. We hope the results of our study will shed more light on the problem of blow-up in Eulerian hydrodynamics.

The results of our work are the following. The linear analysis of small symmetrical deviations from a stationary solution is performed for a pair of anti-parallel vortex filaments and an analog of the Crow instability is found at small wave-numbers. A local approximate Hamiltonian is obtained for the nonlinear long-scale dynamics of this system. Self-similar solutions of the corresponding equations are found analytically. They describe the formation of a finite time singularity, with all length scales decreasing like $(t^* - t)^{1/(2-\alpha)}$, where $t^*$ is the singularity time.

# References


[1] Ruban, V.P., Podolsky, D.I.& Rasmussen J.J.(2001). Finite time singularities in a class of hydrodynamic models. *Phys. Rev. E*, accepted; `arXiv:physics/0012007`.

[2] Kuznetsov, E.A. & Ruban, V.P. (2000). *Phys. Rev. E* **61**, 831.

[3] Ruban, V.P. (1999). *JETP* **89**, 299.

[4] Ruban, V.P. (2000). Phys. Rev. E **62**, 4950.