On helicity conservation laws

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In many cases the motion of covariant derivative used in tensor calculus can be replaced by the notion of exterior derivative, which is more natural and easier to use (especially when the curve-linear coordinates are used). It appears that the equations of motion of various fluids (e.g. ideal fluid, superfluid helium, magnetohydrodynamics) can be formulated in terms of exterior forms and exterior derivatives. It appears also that such a formulation is very convenient for further processing as for instance derivation of conservation laws (conserved currents etc.). The special attention will be payed to the derivation of helicity conservation laws for above mentioned fluids.

For example, if $A = -(\frac{1}{2}\,v^2 + p)\,dt + \vec{v} \cdot d\vec{x}$ – action 1-form and $J = \omega + (v \rfloor \omega) \wedge dt$ is the vorticity current 2-form ($\omega = \omega_{\mu\nu} dx^\mu \wedge dx^\nu$ is the vorticity) then the Euler equation can be written as

$$dA = J,$$

where d is the exterior derivative. The helicity current $\mathcal{H}$ is then defined as

$$\mathcal{H} = A \wedge dA.$$

One easily proves that $d\mathcal{H} = 0$ which implies the conservation of helicity.