On Cauchy problem for quasi-linear singular parabolic equations with singular potentials

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We will use the following notations:
$\mathbf{R}^n_+ \stackrel{\text{def}}{=} \Big\{x \in \mathbf{R}^n \Big| x_n > 0\Big\}$;
$B_+(r) \stackrel{\text{def}}{=} \Big\{x \in \mathbf{R}^n_+ \Big| |x| < r\Big\}$;
$L_{\infty,p}(\Omega) \stackrel{\text{def}}{=} \Big\{f \Big| \operatorname{vrai\,sup} |x_n|^p |f| < \infty\Big\}$;
$\Delta_{\mathcal{B}} \stackrel{\text{def}}{=} \Delta + \frac{k}{x_n^k}\frac{\partial}{\partial x_n}$ with a positive k.

The following problem is considered:

$$\frac{\partial u}{\partial t} = \Delta_{\mathcal{B}} u + \frac{a}{u}|\nabla u|^2 + \Big[b(t) + \frac{d}{x_n^2}\Big]u;\ x \in \mathbf{R}^n_+, t > 0 \tag{1}$$

$$u\Big|_{t=0} = u_0(x);\ x \in \mathbf{R}^n_+ \tag{2}$$

$$\Big(x_n^\alpha \frac{\partial u}{\partial x_n} + \frac{\alpha}{a+1} x_n^{\alpha-1} u\Big)\Big|_{t\to+0} = 0;\ x' \in \mathbf{R}^n, t > 0 \tag{3}$$

Here α is any of the roots of the equation $\alpha^2 + (1-k)\alpha + d(a+1) = 0$ belonging to $(-\infty, \frac{k}{2})$ (at least one such root exists under above assumptions about k, a, d), $b \in C(0, +\infty)$, $a > -1, d \leq \frac{(k-1)^2}{4(a+1)}$, u_0 is non-negative and belongs to $C(\mathbf{R}^n_+) \cap L_{\infty,\frac{\alpha}{a+1}}(\mathbf{R}^n_+)$.

The following assertions are proved:

Theorem 1. There exists a unique classical solution of (1)–(3) belonging to $L_{\infty,\frac{\alpha}{a+1}}(\mathbf{R}^n_+ \times (0, +\infty))$ and this solution is positive.

Theorem 2. If $\int\limits_0^\infty b(t)dt = -\infty$ then for any $x \in \mathbf{R}^n_+$ $\lim\limits_{t\to\infty} u(x,t) = 0$.

Theorem 3. If $\int\limits_0^\infty b(t)dt = B \in \mathbf{R}^1$ then for any $A \geq 0$ for any $x \in \mathbf{R}^n_+$

$$\lim_{t\to\infty} u(x,t) = \frac{A}{x_n^{\frac{\alpha}{a+1}}} \text{ if and only if}$$

$$\lim_{r\to\infty} \frac{(n+k-2\alpha)\Gamma(\frac{n+k}{2} - \alpha)}{\pi^{\frac{n-1}{2}}\Gamma(\frac{k+1}{2}) r^{n+k-2\alpha}} \int\limits_{B_+(r)} x_n^{k-\alpha} u_0^{\alpha+1(x)} dx = A^{\alpha+1} e^{(a+1)B}.$$