On motion of a double helical vortex in a cylindrical tube

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New approach is developed for the velocity estimation of a double helical vortex motion in a cylindrical tube as well as in boundless space.

The phenomenon of double helical vortex is known for a long time. First of all such vortex is being generated in the wake of two-blade propeller or turbine. Double helix represents one of the possible vortex states after vortex breakdown [2]. A perfect double helix rotates and moves translationary without change of his form. The problem on determination of the vortex velocity and frequency of velocity pulsations induced remains actual one. For the vortex in a cylindrical tube there arise additional interesting problem – to find parameters corresponding to immobile double helix. There are known few approaches to the determination of the double helix velocity. Takaki & Hussain [5] gave a formula in a long-wave approximation using the cut-off technique. Wood [6] analyzed the case of small pitch, p, and showed that velocity has an asymptote $\Gamma/2\pi l$ for pair of vortices shifted circumferentially by π radians and with equal circulations Γ. New impact to problem was done by Kuibin & Okulov [3] who used the technique of singularities separation from the Kapteyn series to find velocity of a single helical vortex. This approach allows to avoid the difficulties in the integration of fast oscillating functions. In particular in article cited authors found in limit cases of small and large pitch that regular remainder of the velocity expansion in vicinity of a vortex filament differs by 1/4 from the remainder presenting in the formula for the self-induced rotation of a vortex with finite circular core over which the vorticity is distributed uniformly. Recently Boersma & Wood [1] proved that this difference equals exactly 1/4 for arbitrary vortex pitch. The present work develops the method of singularities separation on the case of a double helix.

The technique of the operating with the Kapteyn series (which present infinite sums of type $\sum\limits_{m=1}^{\infty} mI_m^{(p)}(mx)\,K_m^{(q)}(my)\cos(m\varphi)$, where I_m and K_m are modified Bessel functions, indices p and q denote derivatives of p-th and q-th order) lies in subtracting from modified Bessel functions their uniform expansions and forming summable series. This way allows separating both the pole and logarithmic singularities in the velocity induced by helical vortex filament explicitly.

As the result of analysis for a system of two identical helices shifted by half-period relative each other, with circulation Γ, radius a, pitch $2\pi l$ and the radius of core (with uniform vorticity distribution) ε, the velocity of translational motion along z-axis looks as follows

$$w_z = w_0 + \frac{\Gamma}{2\pi l}\left\{\frac{\tau}{2\left(1+\tau^2\right)^{3/2}}\left[\ln\left(\frac{2\tau}{\sqrt{1+\tau^2}}\frac{a}{\varepsilon}\right)+\frac{3}{4}+\tau^2\right]-H-H^*\right\}.$$

Here w_0 is the value of the axial velocity at the z-axis, parameter $\tau = l/a$ denotes the dimensionless torsion of a helix (the ratio of the torsion to the curvature) or the dimensionless helix pitch. When considering the double helix in an unbounded space the quantity H^* equals zero and $w_0 = \Gamma/2\pi l$ providing vanishing of the induced velocity far from the vortex. The quantity H is a function of a single parameter τ:

$$H = \sum_{m=1}^{\infty} \left\{ \frac{8m}{\tau} K'_m \left(\frac{2m}{\tau}\right) I_m \left(\frac{2m}{\tau}\right) + 2 + \frac{1}{2m} \frac{\tau}{(1+\tau^2)^{3/2}} \right\}.$$

This series converges relatively quickly. Nonetheless as was shown by Kuibin & Okulov [3] there exist method to accelerate its convergence. An asymptotical analysis yields negligible impact of H to the velocity at large helix pitch,

$$H = \left(\frac{1}{2} - \ln 2\right) \tau^{-2} + O\left(\tau^{-4} \ln \tau\right) \quad (\tau \to \infty)$$

In a limit of small pitch it can be also neglected

$$H = -\frac{3\zeta(3)}{64} \tau^3 + O\left(\tau^5\right) \quad (\tau \to 0).$$

The maximum impact of H for $a/\varepsilon = 10$ amounts about 1% .

In the case of double helical vortex in a cylindrical tube of radius R the quantity H^* reads

$$H^* = \frac{\tilde{R}^2}{1-\tilde{R}^4} - \frac{1}{12}\left[\frac{9\eta + 2\eta^3}{(1+\eta^2)^{3/2}} - \frac{3\tau + 2\tau^3}{(1+\tau^2)^{3/2}}\right] \ln\left(1 - \tilde{R}^4\right)$$

$$\tilde{R} = \exp\left[\sqrt{1+\tau^2}\Big/\tau - \sqrt{1+\eta^2}\Big/\eta\right] \left(\sqrt{1+\tau^2} - \tau\right) \Big/ \left(\sqrt{1+\eta^2} - \eta\right), \quad \eta = l/R.$$

When a/R is close to 1 (this is possible only for thin enough cores) H^* dominates in the velocity of vortex motion. It is obvious that for any values of R, Γ, a, l and ε there exist such value of w_0 that the double helical vortex will be immobile.

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