Intensive and weak mixing in the chaotic region of velocity field

Alexandre A. GOURJII
Institute of Hydromechanics, National Academy of Science,
8/4 Zhelyabov str., 01057, Kiev, UKRAINE

The mixing of a passive fluid domain with arbitrary borders in known velocity field is discussed. The method of Local Stretching Maps for various contours is proposed for analyzing the deformation of borders of passive impurity in an arbitrary two-dimensional velocity field. This analyze is explored in a sample of an advection problem of a passive domain in the velocity field induced by a system of point vortices moved periodically (chaotic regime). Investigation shows that the regions of chaotic motion of fluid particles and regions of intensive stirring do not coincide. Chaotic regions have zones of weak stirring, in which contours are transported from one intensive stretching zone to another without any deformation in time.

An adevction problem is limited to the analysis of the trajectories of Lagrangian fluid particles, which form borders of the region under investigation, in Eulerian velocity field. Every fluid particle can be treated as a passive fluid particle. Governing equations of the problem are the system (Cauchy problem) [1, 2].

$$\frac{\partial \boldsymbol{r}}{\partial t} = \boldsymbol{V}(\boldsymbol{r}, t) \qquad \boldsymbol{r}(0) = \boldsymbol{r}^0, \tag{1}$$

Here $\boldsymbol{r}(t)$ is the position of fluid particles (markers) and $\boldsymbol{V}(\boldsymbol{r}, t) = \boldsymbol{V}[U(x, y, t), V(x, y, t)]$ is the velocity field (to be given a priori). Then the investigation is reduced to the study of an evolution of fluid particles in the flow. Every such particle moves along its own trajectory, and the equations (1) can predict particle positions at any moment in time. Ordered connection of markers results to forming domain borders for a given moment.

Our objective reduces to an investigation of the local properties of stretching of various segments and contours initially placed in a neighborhood of an arbitrary point of velocity field. We introduce the stretching maps displayed how initially compact contours placed at current points (x, y) increases its length during a next shot time interval. The ordered set of maps permits to illustrate positions of regions displayed an intensive and weak stirring without direct numerical simulation of an advection problem.

The analyze is based on an solution of eq.(1), which can be written in the linearized form near an arbitrary point (x_0, y_0) for a current moment t_0

$$\begin{cases} d\eta_1/d\tau = a\eta_1 + b\eta_2 + e\tau + g \\ d\eta_2/d\tau = c\eta_1 + d\eta_2 + f\tau + h \end{cases} \quad \text{with} \quad \begin{cases} \eta_1(0) = \eta_1^0 \\ \eta_2(0) = \eta_2^0 \end{cases} \tag{2}$$

where
$$\eta_1 = x - x_0, \quad \eta_2 = y - y_0, \quad \tau = t - t_0, \quad a = \partial U/\partial x, \quad b = \partial U/\partial y,$$
$$c = \partial V/\partial x, \quad d = \partial V/\partial y, \quad e = \partial U/\partial t, \quad f = \partial V/\partial t. \quad g = U_0 \quad h = V_0$$

Here U_0 and V_0 are the constant component of velocity field at the point. The solution of eq.(2) case can be presented in the general form

$$\eta_1(\tau) = \exp(p_1\tau)\,(A\eta_1 + B\eta_2 + E) \qquad \eta_2(\tau) = \exp(p_1\tau)\,(C\eta_1 + D\eta_2 + F) \tag{3}$$

where p_1 is a root of characteristics equation for system (2), which has the largest real part. Functions A, B, C, D, E and F are determined by gradients of the velocity field evaluated at the point (x_0, y_0), and by initial values η_1^0 and η_2^0. Our primary interest is in the length of the vector $|\boldsymbol{\eta}(\tau)| = (\eta_1^2(\tau) + \eta_2^2(\tau))^{1/2}$ in time.

We consider the stretching of smooth contours, which can be presented by a set of passive markers. Their coordinates can be written in a parametrical form

$$x(0, \phi) = |\boldsymbol{\eta}^0| f_1(\tau, \phi) + x_0 \qquad y(0, \phi) = |\boldsymbol{\eta}^0| f_2(\tau, \phi) + y_0 \tag{4}$$

where ϕ is parameter of the contour, and ranges from 0 to ϕ_0; $|\boldsymbol{\eta}|$ is a scale of the figure, the length of which is determined by the integral

$$l(\tau) = |\boldsymbol{\eta}| \int_0^{\phi_0} \left[(\partial f_1/\partial\phi)^2 + (\partial f_2/\partial\phi)^2 \right]^{1/2} d\phi \tag{5}$$

Let us consider a fluid flow generated by a system of point vortices. The motion of N point vortices of strength k_α at position (x_α, y_α) is described by the system of differential equations

$$\dot{z}_\alpha^\star = \frac{1}{2\pi j} \sum_{\beta=1}^{N}{}' \frac{k_\beta}{z_\alpha - z_\beta}, \qquad \alpha = 1, ..., N \tag{6}$$

with initial conditions $z_\alpha(0) = z_\alpha^0$. Here $z_\alpha = x_\alpha + j y_\alpha$, the dot denoted the derivative with respect to time, the asterisk denotes complex conjugate and the prime denotes omission of the singular term $\alpha = \beta$. The equation of motion of a passive Lagrangian fluid particle under the velocity field induced by the system of point vortices may be obtained by considering a marker at $Z_\alpha = X_\alpha + jY_\alpha$ as a point vortex of zero intensity [3].

Consider the periodic motion of three point vortices with the following nondimensional initial condition: $k_1 = k_2 = -k_3 = 1.0$, $x_2^0 = -x_1^0 = 0.497...$, $x_3^0 = 0.0$, $y_1^0 = y_2^0 = -2.0$, and $y_3^0 = -4.0$. Reference [3] provides the details of this type of interaction.

All global criteria used in the analyses (Poincare section, Largest Lyapunov exponent, phase trajectories of some passive fluid particles, spectrum analyses) testify that there are regular and chaotic zones of motion. The Poincare section allows us to indicate global regions of intensive stretching (chaotic motion of passive fluid markers).

Detail investigation and comparison analyze of achieved results for various types of initial contours with direct numerical simulation of an advection process show: the evolution of the contours depends on the velocity field induced by vortices. If, during an advection process, some segment of a contour is far enough from the vortices, it does not undergo a strong deformation and moves with almost the same form and size, even if it is in the chaotic domain of Poincare section.

Construction of local stretching maps for various contours for the fixed moment has a considerable utility in a preliminary analysis of the advection process of passive fluid contours in velocity field to be given. These maps help not only to find the existence of regions of intensive stirring, but also, using the set of maps, to detect their drift in time.

The example of passive advection in the velocity field of three point vortices allowed us demonstrate this method to detect the domains of intensive stirring. Our analysis shows that regions of weak stirring exist in the chaotic domains of passive fluid motion. These regions move in time but remain contained inside the chaotic region of fluid motion. Presented method identifies the regions of strong stirring and estimates the stretching effects in these regions without direct numerical simulation of an advection problem.

References

[1] Ottino J.M., *"The Kinematics of Mixing: Stretching, Chaos and Transport"*, Cambridge University Press, 1989.

[2] Aref H., Phil. Trans. R. Soc. London, 333 (1990) 273.

[3] Meleshko V.V., and Gourjii A.A, Stirring of an inviscid fluid by interacting point vortices // in "Modelling of Oceanic Vortices" (ed. G.J.F. van Heijst), Amsterdam, Noth-Holland (1994) 271.