A study of singularity formation in a class of solutions of the Euler & ideal MHD equations

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This work (Ohkitani and Gibbon [2]) concerns a class of solutions of the three-dimensional incompressible Euler equations of the type

$$\boldsymbol{U}(x,y,z,t) = \{u_1(x,y,t),\, u_2(x,y,t),\, z\gamma(x,y,t) + W(x,y,t)\}$$

which evolve on a tubular domain, infinite in the z-direction with periodic cross-section $\mathcal{A} = [0,L]^2$. It is based on a decomposition of the three-dimensional Navier-Stokes and Euler equations, found by Gibbon *et al* [1], into two-dimensional equations for γ, W and the third vorticity component $\omega = u_{2,x} - u_{1,y}$. A numerical study of these indicates the existence of a finite time singularity in γ and other variables. This is an infinite energy blow-up on an infinite domain and therefore belongs to a different category than finite energy Euler singularities. The three dimensional vortices that appear within the tube just prior to blow up have a flower-like structure, with petals of high vorticity interspersing hollow regions of weak vorticity. Constantin [3] has subsequently proved analytically that this singularity exists and that it is two-sided; that is, $\gamma \to \pm\infty$ in different parts of $\mathcal{A}$.

For three-dimensional ideal MHD (Gibbon and Ohkitani [4]) using Elsasser variables $\boldsymbol{V}^{\pm} = \boldsymbol{U} \pm \boldsymbol{B}$ we investigate a similar class of solutions: $\boldsymbol{V}^{\pm} = \left(\boldsymbol{v}^{\pm},\, v_3^{\pm}\right)$ and $\boldsymbol{v}^{\pm} = \boldsymbol{v}^{\pm}(x,y,t)$ with $v_3^{\pm}(x,y,z,t) = z\gamma^{\pm}(x,y,t) + \beta^{\pm}(x,y,t)$. Two-dimensional PDEs for $\gamma^{\pm}$, $\boldsymbol{v}^{\pm}$ and $\beta^{\pm}$ are obtained, valid in the tubular domain $\mathcal{A}$. Pseudo-spectral computations provide evidence for a finite time blow-up. This apparent blow-up is an infinite energy process that gives rise to certain subtleties; while all the variables appear to blow-up simultaneously, the two-dimensional part of the magnetic field $\boldsymbol{b} = \frac{1}{2}(\boldsymbol{v}^{+} - \boldsymbol{v}^{-})$ blows up very late. This singularity in $\boldsymbol{b}$ is hard to detect numerically but supporting analytical evidence of a Lagrangian nature is provided for its existence. In three dimensions these solutions correspond to magnetic vortices developing along the axis of the tube that open prior to breakdown.

References

[1] Gibbon J. D., Fokas A. & C. R. Doering (1999). Dynamically stretched vortices as solutions of the Navier-Stokes equations. *Physica D* **132**, 497.

[2] Ohkitani K. & Gibbon J. D. (2000). Numerical study of singularity formation in a class of Euler and Navier-Stokes flows. *Phys. Fluids* **12**, 3181–3194.

[3] Constantin P. (2000). The Euler equations and non-local conservative Riccati equations. *Internat. Res. Notices (IMRN)* **9**, 455.

[4] Gibbon J. D. & Ohkitani K. (2000). Numerical study of singularity formation in solutions of the equations for incompressible MHD. *Preprint.*