Linear stability of a vortex ring revisited

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Vortex rings are invariably susceptible to wavy distortions, sometimes leading to disruption. We revisit the development of waves on vortical cores, namely, the linear stability problem. It is widely accepted that the *Widnall instability* is responsible for development of unstable waves. This is an instability for a straight vortex tube subjected to a straining field in a plane perpendicular to the tube axis (Moore & Saffman [1], Tsai & Widnall [2]).

When viewed locally, a thin vortex ring is crudely approximated by a straight tube. For simplicity, we restrict our attention to a circular core of uniform vorticity, that is, the *Rankine vortex*. The waves of infinitesimal amplitude on the Rankine vortex, the *Kelvin waves*, are known to be neutrally stable. The vortex ring induces, on itself, not only a local uniform flow but also a local straining field akin to pure shear [3]. A pure shear with the principal axes perpendicular to the vorticity deforms the core into an ellipse. This is a quadrupole field proportional to $\cos 2\theta$ or $\sin 2\theta$, in terms of polar coordinates (r, θ) in the meridional plane, and is capable of mediating a parametric resonance between the bending waves of left- and right-handed. In a wider context, straining field may be looked upon as the flow induced by neighbouring vortices. Bayly [4] and Waleffe [5] uncovered that the *elliptical instability* has much in common with the Widnall instability, establishing ubiquity of the latter.

Recently Fukumoto & Moffatt [6] have made an attempt at calculating, with a high accuracy, the translation speed and the expansion, in toroidal radius, of an axisymmetric vortex ring in a viscous fluid. Using the method of matched asymptotic expansions in a small parameter ϵ, the ratio of the core to the ring radii, the solution of the Navier-Stokes equations is built in a series form in powers of ϵ. The leading-order flow is a circular vortex, a monopole field. The first-order flow is a *dipole* field proportional to $\cos\theta$, and the quadrupole field appears at the next order. Though the dipole field, pertaining to the curvature effect, plays the key role, its influence on stability has gone untouched. In the present study, we explore "whether the dipole field at $O(\epsilon)$ can cause instability or not".

According to Krein's theory of parametric resonance in Hamiltonian systems [7], a single Kelvin mode cannot be destabilized by perturbations breaking the circular symmetry. But an instability becomes permissible for a superposition of two modes with the same wavenumber and the same frequency. Subjected to the dipole field as the dominant symmetry-breaking perturbation, two Kelvin modes with angular dependence $e^{im\theta}$ and $e^{in\theta}$ can be amplified at the intersection points of dispersion curves if the condition $|m-n| = 1$ is met.

We show that a parametric resonance indeed occurs at a part of intersection points of dispersion curves of *axisymmetric* ($m = 0$) and *bending* ($n = 1$ or $n = -1$) modes. The growth rate is evaluated and is compared with that of the Widnall instability [3]. It is found that the dipole effect predominates over the quadrupole effect for very thin cores. The instability mechanism is clarified from the viewpoints not only of flow structure but also of the Hamiltonian spectrum theory. A comment is given to the cases of continuous distribution of vorticity (*cf.* Kop'ev & Chernyshev [8]).

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