Asymptotic Structure of Fast Dynamo Eigenfunctions

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Hydrodynamic and hydromagnetic instabilities provide mechanisms for the development of complexity in fluid flows. An instructive prototype is the kinematic dynamo problem, in which the velocity field of an electrically-conducting fluid is prescribed and magnetic field eigenmodes with positive growth rates are sought. Numerical simulations give compelling evidence that in many flows exhibiting Lagrangian chaos, plus a sufficient degree of "non-cancellation", the growth rate remains bounded above zero in the limit of zero magnetic diffusivity. The corresponding eigenfunctions, however, exhibit more and more complicated spatial structure as the diffusivity goes to zero, converging (if at all) only to a distribution defined in the sense of generalized functions. The limiting distribution may be thought of as a singularity occupying a finite volume of space.

Other hydrodynamic and hydromagnetic instability problems have the same mathematical structure as the fast dynamo problem, and exhibit similar behavior in the limits of vanishing viscous and magnetic diffusivity. The structures observed in the development of complexity in real flows are expected to reflect the asymptotic structures of these singular eigenfunctions. Obtaining an analytical description of these structures is a difficult problem, however, especially

when the base flow exhibits Lagrangian chaos.

This paper considers the situation in which the base flow is a superposition of a small number of sinusoids with integer wavenumber components. Such flows (e.g. the ABC flow) have been used extensively in numerical investigations of fast dynamo action. In Fourier space the eigenvalue problem becomes a multidimensional difference equation in which each Fourier mode is linked to a small finite number of its neighbors. Although the problem is now discrete, when the magnitude of the wavevector becomes large

the amplitudes become slowly-varying in a WKB sense, and the difference equation can be approximated by a hierarchy of differential equations in wavevector space. Moreover, since the Laplacian becomes scalar multiplication in wavevector space, it is relatively straightfoward to consider the effects of small finite diffusivity versus exactly zero diffusivity.

For zero diffusivity, the leading order equations reproduce advection and distortion of localized wavepacket disturbances in physical space, and the first correction reproduces stretching and twisting by the physical

strain field. The 'source' of the wavepackets is the low-wavevector Fourier modes of the eigenfunction, whose amplitudes cannot be determined within the high-wavevector theory. The eigenvalue, i.e. growth rate, also cannot be determined within the high-wavevector theory. There is no decay of amplitude as wavevector increases, which reflects the presumably singular nature of the zero-diffusivity eigenfunctions.

When the diffusivity is small and finite the leading order theory remains the same, but the first correction contains an additional term reflecting decay of the wavepacket as it advects, distorts, stretches, and twists. This decay regularizes the high-wavenumber behavior and removes the singularity of the eigenfunction. The eigenvalue also receives a correction due to small finite diffusivity. Although the leading-order eigenvalue is determined by the low-wavevector modes (and is inaccessible by the present technique), the diffusive correction is determined by the high-wavenumber modes and can be calculated within the asymptotic theory.

Thus the asymptotic theory confirms and supports the picture that has emerged from numerical simulations at moderately small diffusivities, and indicates that the same behavior is likely to continue in regimes beyond the capabilities of numerical computation.