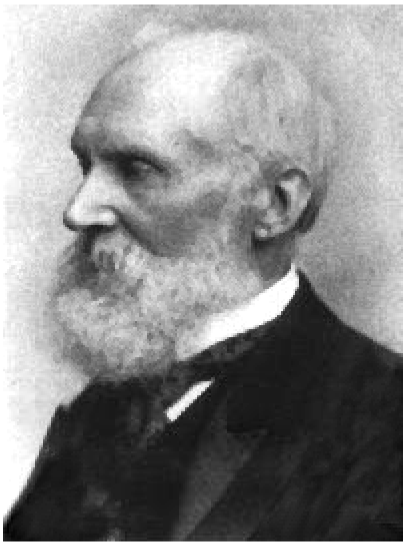
Stokes and Kelvin, a century later: an essay

Keith Moffatt [1]

George Gabriel Stokes



William Thomson Kelvin

The year 2003 marked the centenary of the death of George Gabriel Stokes (1819-1903), and a meeting was held in Cambridge, where he was Lucasian Professor of Mathematics for more than half a century (1849-1903), to commemorate his life and work. I was asked to lecture on Stokes's contributions to fluid mechanics and I focussed particularly on his role as the pioneer of the dynamics of real (*i.e.* viscous) as opposed to 'ideal' fluids. It may be of interest to readers of this Newsletter if I reproduce parts of this lecture here, but I shall also intersperse this with reflections of a more personal nature.

Stokes's name will of course forever be coupled with that of Navier through the governing equations of fluid mechanics; but it is also permanently attached to the concept of Stokes flow (the viscous limit in which inertia forces may be neglected) initiated through his seminal study of the flow past spheres and cylinders.

Stokes's career was inextricably linked with that of William Thomson (Lord Kelvin) (1824-1907), his lifelong friend and correspondent; plans will no doubt soon be afoot to mark Kelvin's centenary also, both in Glasgow where he was Professor of Natural Philosophy (and this also for more than half a century 1846-1899), and in Cambridge where he was an undergraduate (1841-1845) and where he frequently sojourned as a Fellow of Peterhouse (1845-52 and 1872-1907) during his subsequent phenomenal career. His final visit was in

[1] Trinity College, Cambridge, United Kingdom

February 1903 when he stood by the graveside of Stokes and is alleged to have uttered the words: “Stokes is dead; I shall visit Cambridge no more”. G.I.Taylor told me once at lunch in Trinity back in the 60s that he had attended Kelvin’s 1904 lecture to the British Association for the Advancement of Science; this is my slender personal connexion with these great pioneers of 19th century science!

In reflecting upon the lives of Stokes and Kelvin, I have been struck by certain parallels that can be drawn between them and two great figures of our own recent era, namely George Keith Batchelor (1920-2000) and Michael James Lighthill (1924-1998), whose careers resonate in numerous respects, albeit almost exactly one century later, with those of Stokes and Kelvin respectively. Like these two, Batchelor and Lighthill interacted for many years somewhat like sparring partners, enlivened by the tensions necessarily associated with their strongly divergent personalities. We were privileged in Cambridge to have these great scientific personalities in our midst, a presence that did so much to shape the development of fluid dynamics throughout the latter half of the 20th century, not only at the parochial Cambridge level, but also, through their widespread national and international influence, on the global stage.

The passage of time allows us to view great scientists of the past in their historical context and to better appreciate the scope and magnitude of their achievements. So it has been with Stokes and Kelvin, and particularly so for those who work in fluid mechanics, a subject that was influenced and shaped in so many ways by the brilliance of their investigations. David Wilson’s (1987) comparative study *“Kelvin and Stokes”*, and his edition of *“The Correspondence between Sir George Gabriel Stokes and Sir William Thomson, Baron Kelvin of Largs”* (1990) have cast penetrating light on the interactions between these great men of science; I have drawn freely on these works in the following discussion. The papers of Stokes to which I refer may be found in his Collected *Mathematical and Physical Papers* (Stokes 1905).

Stokes was born in Co. Sligo in Ireland, son of the vicar of Screen, in which village a meeting is now held every three years or so, commemorating aspects of Stokes’s life and work. I attended one of these meetings in 1998; it was held in the classroom of the primary school that Stokes had attended as a child; we used a blackboard mounted on its easel as our primary visual aid!

Stokes moved to England in 1835 and studied at Bristol College (precursor of Bristol University) for two years before coming to Cambridge as an undergraduate in 1837. He studied the Mathematical Tripos (so-named because of the medieval practice whereby students underwent the oral examination in mathematics while seated upon a three-legged stool). Students who attained first-class Honours in the Mathematical Tripos were (and indeed still are) known as Wranglers (to ‘wrangle’ being to engage in disputatious argument), and the Wranglers were formerly placed each year in order of merit, the first on the list being accorded the coveted title of 'Senior Wrangler', a distinction that Stokes duly attained in 1841, the same year in which the younger William Thomson (at age 17) was admitted as an undergraduate of Peterhouse, the most ancient of Cambridge's many Colleges. On the basis of his success, Stokes was immediately elected to a Fellowship at Pembroke College.

It is interesting to note that, although the word Wrangler is used to this day in Cambridge, the listing is now alphabetical rather than by order of merit. The change was adopted in 1909 (G.I.Taylor who graduated 22nd Wrangler in 1907 was one of the last to suffer the slings and arrows of numerical listing!), on the grounds that the publication of an order-of-merit had an undesirable tendency to breed intense and potentially unhealthy competition. (I note that no such scruples are evident in that other great teaching establishment of which I had experience in the 1990s, the Ecole Polytechnique in Palaiseau, France, where the list of graduating students was still published each year in order-of-merit all the way from the first on the list to the four-hundredth!)

From the start, Stokes was conscious of the wide divergence between the predictions of the classical theory of irrotational flow and the results of common observational experience. In his 1843 paper *"On some cases of fluid motion"*, he sought to confront this divergence head-on: he wrote in the following terms:

"The only way by which to estimate the extent to which the imperfect fluidity [viscosity] of fluids may modify the laws of their motion, without making any hypothesis as to the molecular constitution of fluids, appears to be, to calculate according to the hypothesis of perfect fluidity some cases of fluid motion, which are of such a nature as to be capable of being accurately compared with experiment."

One of the cases studied in this paper was the flow of fluid (assumed irrotational) in a closed box whose interior is of the form of a rectangular parallelipiped, the box being subjected to an arbitrary rigid-body motion. Stokes solved this problem by adroit use of Fourier series. One may easily carry out an experiment (and Stokes probably did), by suspending a transparent box filled with water on a torsion wire and subjecting it to torsional oscillations; unless the amplitude of these oscillations is extremely small, the flow (which may be visualised using suspended particles – one may use tea-leaves, as Stokes might well have done – he was a great tea-drinker!) is different from the 'perfect-fluidity' potential flow, and most obviously so in the boundary layers (later to be called Stokes layers) that form on the interior surface of the box. I shall comment further on this flow below.

Stokes's great paper *"On the theories of the internal friction of fluids in motion, and of the equilibrium and motion of elastic solids"* was published in 1845. In the course of his introduction he refers to a previous derivation of what we now refer to as the Navier-Stokes equations by Poisson, and he adds a footnote "The same equations have also been obtained by Navier (*Mém. de l'Académie*, t.vi. p.389), but his principles differ from mine still more than do Poisson's ". Navier had assumed a very specific model involving particles at the points of a lattice and subject to interactive forces linearly related to their instantaneous relative velocities – what we may now recognise as the first construction of a 'lattice gas dynamics'. Stokes, by contrast, sought to develop a theory based on the concept of a continuum, and freed from any assumption concerning the molecular structure of the fluid. He developed the concepts of stress and rate-of-strain and assumed these to be related in linear (Newtonian) manner, leading to the now familiar N-S equations. It is interesting to note that Stokes was aware of the problem of 'dilitational viscosity', and wrote "The equations at which I have thus arrived contain two arbitrary constants, whereas Poisson's equations contain but one". The power and generality of the approach pioneered by Stokes is evidenced by the fact that this same approach is almost invariably used nowadays in any treatment of the fundamentals of fluid dynamics.

Five years later came Stokes's paper *"On the effect of the internal friction of fluids on the motion of pendulums"*, and here it is interesting to note that it was indeed the practical problem of determining the effect of air friction on the damping of a pendulum that motivated this pioneering study. Stokes discusses first the neglect of the nonlinear **u**.grad **u** term of the N-S equations, and second, what we would now describe as the dynamical-similarity properties of the resulting linearised equations; he then presents his solution for the oscillatory layer at a boundary oscillating parallel to itself (now called the Stokes layer), and only then turns his attention to the problem of the flow due to an oscillating sphere (like the bob of a pendulum undergoing small-amplitude oscillations). He solves this problem completely, then as a postscript passes to the steady (zero-frequency) limit, and obtains the famous 'Stokes drag' formula, $F = 6\pi\mu aV$, often described as the best-known result in all fluid mechanics. Not satisfied with this, he turns his attention to the problem of flow past an oscillating cylinder (modelling the flow past the wire supporting the bob of the pendulum), and notes a troubling divergence in this case in his series solution in the zero-frequency limit. He describes this as "a difficulty in the case of a cylinder", a difficulty that was in fact only resolved more than a century later with the development of matched asymptotic expansions (Lagerstrom & Cole 1955, Proudman & Pearson 1957). This 141-page paper of Stokes must surely be one of the greatest in fluid mechanics ever written.

What seems rather extraordinary to me is that (with only a handful of notable exceptions) so little further fundamental advance in the mechanics of viscous fluids was made over the next half-century (1850-1900). Was this because

Stokes himself became preoccupied with other areas (potential theory of surface waves, optics, the mathematics of infinite series, …) and with his exceptionally heavy responsibilities as Secretary of the Royal Society and Editor of its Philosophical Transactions over most of his subsequent active research life (1854-1885)? Or was it perhaps that the development of vortex dynamics pioneered by Helmholtz (1858), and much promoted by Kelvin (*"On vortex atoms"* 1867) as providing a model for the ultimate structure of matter, had an overpowering influence in swinging the pendulum of scientific investigation back towards the Eulerian domain of ideal fluids? For whatever reason, it is a fact that the next fundamental advance in 'real' fluid dynamics had to await Prandtl's (1905) introduction of the boundary-layer concept.

There was however an interesting precursor to Prantdl's boundary-layer theory in which Stokes played a part. This was provided by Hele-Shaw's (1898) experiments on the flow in a narrow gap between two parallel boundaries in which obstacles of various shapes may be placed (the Hele-Shaw cell). Stokes (now aged 79), with characteristic lucidity, wrote an Appendix to this paper entitled *"Mathematical proof of the identity of the streamlines obtained by means of a viscous film with those of a perfect fluid moving in two dimensions".* Again, Stokes's treatment is precisely that which finds its way into most modern textbooks of the subject. The two-scale treatment adopted by Stokes is just what Prandtl would need in his subsequent development of boundary-layer theory.

Stokes's remarkable correspondence with Kelvin extended from 1846 to 1901. Typical is an exchange that took place in the Spring of 1847. On 30th March, Kelvin, already installed in Glasgow, wrote:

"My dear Stokes,
It has just occurred to me this evening that you may possibly be able to give me some information that will help me out of a difficulty which has been puzzling me for a considerable time. …"

He goes on to pose in physical terms a problem concerning potential flow within a bounded domain. Stokes's reply from Pembroke College, Cambridge, is dated 1st April; the speed of communication using the recently established penny-post was remarkable -- every bit as efficient as modern e-mail! Stokes restates Kelvin's problem in more precise mathematical terms, and proceeds to give a preliminary solution. He follows this up with two further long letters dated 3rd and 5th April, in which he gives a wide-ranging discussion of the problem. Kelvin replies to all three letters on 7th April, where he says:

"Many thanks for your letters, which have given me plenty of matter for contemplation, in subjects with which I have long been interested. There is a great deal which I would like to say about them, but I do not know where to begin, especially as I am about to start for Ireland in a few hours, to take advantage of half a week's holiday …"

The exchange was typical of later correspondence also: Kelvin would throw out a plethora of physical ideas ranging over fluid dynamics, electromagnetism, thermodynamics, … ; and Stokes would endeavour to bring more disciplined thinking to bear on these ideas to the point at which they could be properly formulated in mathematical terms. This was truly a symbiotic relationship between two men of quite exceptional and yet complementary talents.

And why do I seek to draw the comparison between Stokes and, in our own era, the late Professor G.K.Batchelor? Like Stokes, Batchelor, having arrived in Cambridge from Australia in 1945, spent the rest of his life there. He was elected to a Fellowship of Trinity College in 1947, and, from 1948 to 2000, was successively Lecturer, Reader, Professor and Emeritus Professor of the University. In 1956, he founded the *Journal of Fluid Mechanics (JFM*), and, just as Stokes had devoted himself to *Phil Trans Roy Soc*, so Batchelor devoted himself to *JFM* for more than four decades. Both men were what could be described as 'supremely conscientious', with a strong personal commitment to the essential morality of science. Both made seminal contributions to fluid mechanics, in Batchelor's case, to the theory of homogeneous turbulence, and later to microhydrodynamics, appropriately the application of Stokes' theory to suspensions of particles, drops or bubbles in fluids. Batchelor was of course a Co-Founder of Euromech in 1966 (I have detailed his many achievements in his Biographical Memoir, Moffatt 2002).

And why do I similarly propose the late Sir James Lighthill as the fluid dynamicist of recent times who most closely mirrors the genius of Kelvin? Like Kelvin, Lighthill showed early signs of genius, qualifying in parallel with his close friend and classmate at Winchester, Freeman Dyson, for a major scholarship to Trinity College, Cambridge, at the exceptionally early age of 15. Like Kelvin, he graduated as a Wrangler in 1943 (the earliest date at which, according to the regulations he could do so). Kelvin had been disappointed to be second Wrangler in 1845; as indicated above, wranglers were not numerically ordered after 1909, but it was nevertheless common knowledge that Lighthill in fact came second to Dyson in the 1943 examination – no doubt a powerful stimulus to prove himself decisively in research in the years that followed! In fact, just as Kelvin had been elected to the Chair of Natural Philosophy at Glasgow at the spectacularly early age of 22, so Lighthill was elected to the Beyer Chair of Applied Mathematics at Manchester at the equally spectacular (for its time) age of 26. Lighthill's brilliant achievements in supersonic aerodynamics and aeroacoustics during the 1950s have been well described by Pedley (2001) and I shall not endeavour to summarise them here. Suffice it to say that they reveal hallmarks of genius that bear comparison with those attributed to Kelvin in relation to his work of the 1850s on the foundations of thermodynamics.

But the comparison does not end here: for just as Kelvin had subsequently devoted immense energy to problems associated with the laying of the first transatlantic telegraph cable, so Lighthill devoted himself during his years as Director of the Royal Aircraft Establishment at Farnborough (1959-1964) to problems associated with the first (and last?) commercial transatlantic supersonic aircraft, Concorde. Furthermore, just as Kelvin showed unbounded ambition and imagination in formulating his fundamental theory of matter – the vortex atom theory referred to above, so Lighthill showed comparably boundless intellectual energy and imagination in seeking to explain evolutionary biology in his later years of research both at Cambridge and at University College London, through a far-reaching investigation of the aerodynamics of flight and the hydrodynamics of swimming of insects, birds and fish, i.e. the greater part of the whole animal kingdom. The comparison is compelling, is it not?

I would like to end this essay on a personal note in relation to Stokes. In the course of my own career, I have worked on three problems within the field of Stokes flow that I would have dearly liked to discuss with him. We perhaps all have our individual favourites in this regard! Here are mine:

First, there is the problem of corner eddies that I described in 1964, and that were beautifully visualised experimentally by Taneda (1979): two-dimensional Stokes flow in a corner generally exhibits an infinite sequence of geometrically and dynamically self-similar eddies, a phenomenon that appears quite astonishing, bearing in mind that Stokes flows under prescribed boundary conditions are flows that minimise the rate of dissipation of kinetic energy. Of course, the eddies decay very rapidly as the corner is approached, and only the first two or three can be detected in experiments. Current work however (Branicki & Moffatt, in preparation) reveals that for time-periodic Stokes flow in a corner, these eddies 'come to life' one by one in a most intriguing way. Stokes's problem of the torsionally-oscillating parallelipiped, to which I have referred above, is well adapted to reveal this behaviour, far removed from that obtained by potential flow analysis!

Second, there is the problem of 'free-surface cusps' on which I worked with Jae-Tack Jeong some years ago (Jeong & Moffatt 1992). Theory based on the steady Stokes equations indicates a quite extraordinary formula for the minimum radius of curvature R on a free surface when viscous effects compete with surface tension effects in determining the free surface shape: if d is a characteristic geometric scale for the problem, then R/d is proportional to exp{-32π *Ca*) where *Ca* is the capillary number, essentially the ratio of the viscous force to the capillary force in the neighbourhood of the free surface. Under a 'level-playing-field' assumption *Ca* = 1, the formula gives a value of R/d of order 10^{-42}! This, so far as I know, is the smallest non-dimensional number to emerge from any problem in continuum mechanics when the input parameters (here just the capillary number) are of order unity. This may certainly be described as a physical (though not a mathematical) singularity. One way to resolve the singularity (Eggers 2001) is to take account of the variable pressure distribution on the (no longer) 'free' surface due to the flow in the cusp region of the air above the surface.

My third choice is the phenomenon of chaos in steady Stokes flows (Bajer & Moffatt 1990). The fact that the streamlines of a steady Stokes flow inside a sphere, driven by a smooth tangential velocity distribution on the spherical surface, can be chaotic, came as quite a surprise! One might reasonably expect that steady Stokes flows, dominated as they are by strong smoothing viscous effects will exhibit maximum regularity. Not so! In three dimensions, they are generically chaotic, in the sense that initially adjacent fluid particles move apart exponentially with time (i.e.the Liapunov exponent is positive). The streamlines are not closed, neither do they lie on surfaces; they inhabit subdomains of the sphere of (no doubt) fractal character.

I like to think that Stokes would have been particularly intrigued by these three problems, all within the branch of the subject that bears his name. I have a similar group of problems that I would dearly like to discuss with Kelvin, but that is another story, that can perhaps best wait until 2007!

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List of References

1. Bajer, K & Moffatt, H.K. 1990 J.Fluid Mech. **212**, 337-364
2. Eggers, J. 2001 Phys.Rev.Lett. **86**, 4290
3. Hele-Shaw, 1898 Report of the British Association (see also Nature, **58,** 34 (1898))
4. Helmholtz, H. 1858 Crelle's *Journal für die reine und angewandte Mathematik*, **55**, 25-55
5. Jeong, Jae-Tack & Moffatt, H.K. 1992 J.Fluid Mech. **241**, 1-22
6. Lagerstrom P.A.& Cole, J.D. 1955 J.rational Mech.Analysis, **4**, 817
7. Moffatt, H.K. 1964 J.Fluid Mech. **18**, 1-18
8. Moffatt, H.K. 2002 George Keith Batchelor. Biogr. Mems. Fell.R.Soc.Lond. **48**, 25-41
9. Pedley T.J. 2001 Sir Michael James Lighthill. Biogr. Mems. Fell.R.Soc.Lond. 47, 333-368
10. Prandtl, L. 1905 Verhandlg.III. Int. Math. Kongr. Heidelberg, 574-584

11. Proudman, I. & Pearson, J.R.A. 1957 J.Fluid Mech. **2**, 237-262

12. Stokes, Sir George Gabriel , Collected Papers, Vols I-IV, Cambridge University Press

13. Taneda, S. 1979 J.Phys.Soc. Jpn. **46**, 1935-1942

14. Thomson,W.(Lord Kelvin) 1867 Phil.Mag. **34,** 15-24

15. Wilson, D.B. 1987 Kelvin and Stokes, A Comparative Study in Victorian Physics. Adam Hilger, Bristol.

16. Wilson, D.B. 1990 The Correspondence between Sir George Gabriel Stokes and Sir William Thomson, Baron Kelvin of Largs, Volume 1, 1846-1869; Volume 2, 1870-1901. Cambridge University Press.